Complexity of the zero set of a matrix Schubert ideal

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T-varieties

A T-variety of complexity-d is an affine normal variety X that admits an effective T-action with

$$\dim(X) - \dim(T) = d.$$

- T-varieties of complexity-0 are *toric varieties*
- Complexity measures how far a T-variety is from being toric.

Matrix Schubert varieties

Consider the action of $B \times B$ on $\mathbb{C}^{n\times n}$ given by

$$(B \times B) \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$$
$$((X, Y), M) \mapsto XMY^{-1}$$

The **matrix Schubert variety** associated to a permutation $w \in S_n$ is the Zariski closure $\overline{X_w} := \overline{BwB} \subset \mathbb{C}^{n\times n}$.

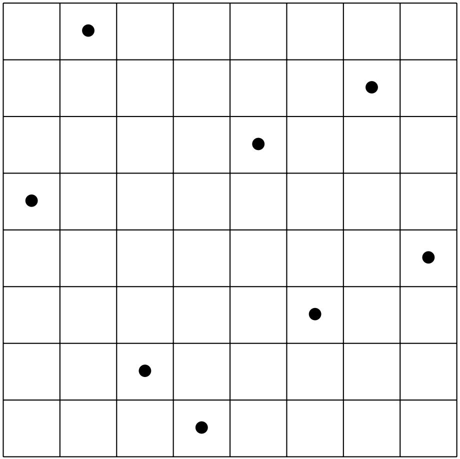
- Torus acting on $\overline{X_w}$ is $T \times T$
- $\overline{X_w} = \{M \in \mathbb{C}^{n\times n} : \mathrm{rk}_M(a, b) \leq \mathrm{rk}_w(a, b) \text{ for all } (a, b) \in [n]^2\}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$w = 51423$

Opposite Rothe diagram

$$D^\circ(w) := \{(i,j) : w(j) < i, w^{-1}(i) > j\}$$



$D^\circ(41783625)$

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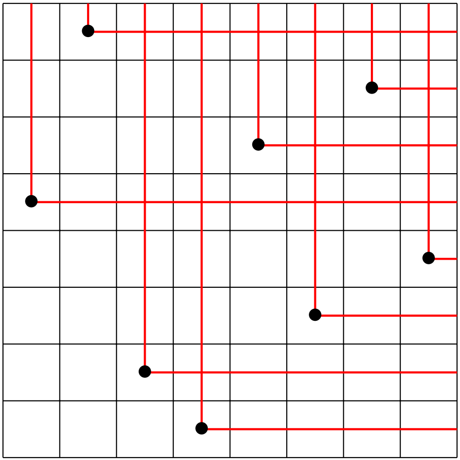

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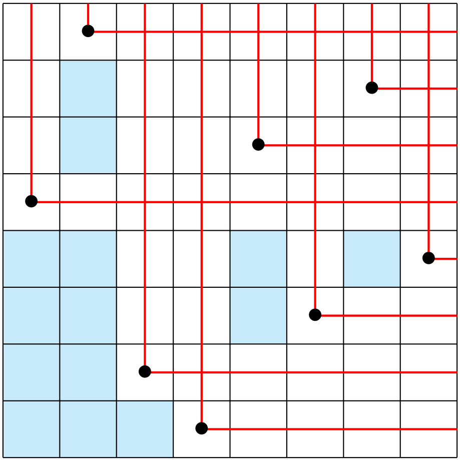
- $|D^\circ(w)| = \#\text{Noninversions of } w$
- Connected components of $D^\circ(w)$ are French Young diagrams



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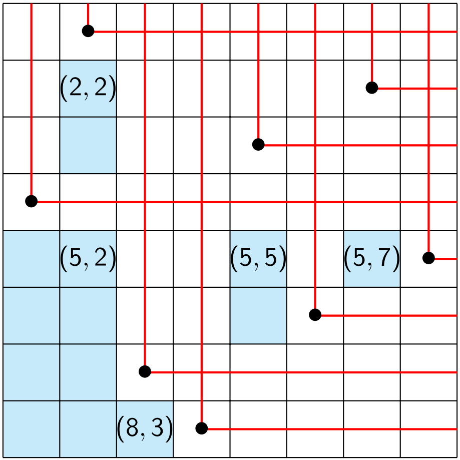
- $|D^\circ(w)| = $ #Noninversions of w
- Connected components of $D^\circ(w)$ are French Young diagrams
- $\mathrm{Ess}(w)$ is the union of NE-corners of each connected component of $D^\circ(w)$



$D^\circ(41783625)$

Fulton's Essential Set Theorem

Theorem (Fulton, '92)

The matrix Schubert variety $\overline{X_w}$ *is an affine variety of dimension* $n^2 - |D^\circ(w)|$ *defined as a scheme by the determinants encoding the inequalities* $\mathrm{rk}_M(a, b) \leq \mathrm{rk}_w(a, b)$ *for all* $(a, b) \in \mathrm{Ess}(w)$.

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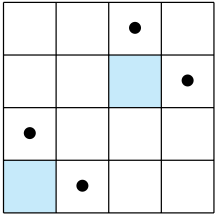
Example

- $\overline{X_{3412}}$ is defined by $\mathrm{rk}_M(4,1) \le \mathrm{rk}_w(4,1) = 0$ and $\mathrm{rk}_M(2,3) \le \mathrm{rk}_w(2,3) = 2$
- $\left(z_{41}, \det \begin{pmatrix} z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \\ z_{41} & z_{42} & z_{43} \end{pmatrix} \right) \subset \mathbb{C}[z_{11}, \ldots, z_{44}]$
- $\dim(\overline{X_{3412}}) = 16 - 2 = 14$

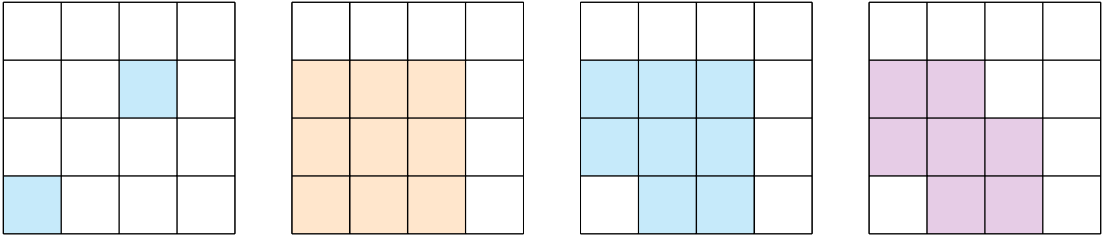


$D^\circ(3412)$

$\overline{X_w} = Y_w \times \mathbb{C}^k$

Every $\overline{X_w}$ can be written as $\overline{X_w} = Y_w \times \mathbb{C}^k$ where k is maximal.

- $\overline{X_w}$ and Y_w have the same defining ideal
- $Y_w :=$ projection of $\overline{X_w}$ onto the entries of $L(w)$
- $\overline{X_w} = Y_w \times \mathbb{C}^{n^2 - |\mathrm{SW}(w)|}$
- $\dim(Y_w) = |L'(w)|$.



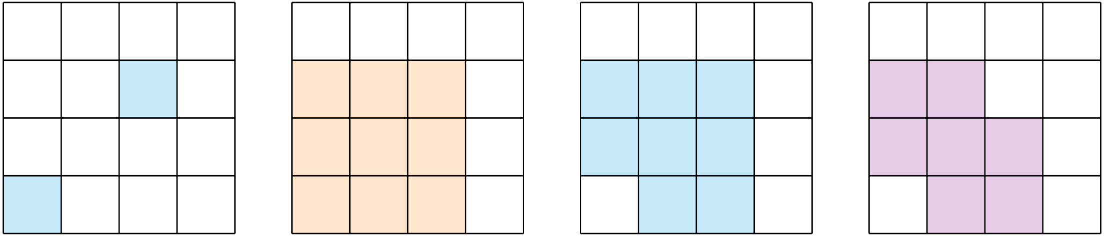
$D^\circ(3412)$ SW(3412) $L(3412)$ $L'(3412)$

Example: $\overline{X_{3412}} = Y_{3412} \times \mathbb{C}^7$

Y_{3412} is the hypersurface defined by the ideal

$$\left(\det \begin{pmatrix} z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \\ 0 & z_{42} & z_{43} \end{pmatrix}\right) \subset \mathbb{C}[z_{11}, \ldots, z_{44}]$$

- $\dim(Y_{3412}) = |L'(3412)| = 7.$
- $n^2 - |\mathrm{SW}(3412)| = 16 - 9 = 7$



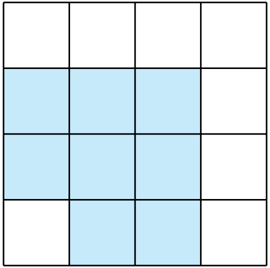
$D^\circ(3412)$ $\mathrm{SW}(3412)$ $L(3412)$ $L'(3412)$

Dimension of $T \times T$

Given $w \in S_n$, let G^w be the acyclic bipartite graph with

- $V(G^w) \subseteq \{1, \ldots, n\} \sqcup \{\overline{1}, \ldots, \overline{n}\}$
- $E(G^w) = \{\, (a \to \overline{b}) : (a, b) \in L(w) \,\}$

such that G^w has no isolated vertices.

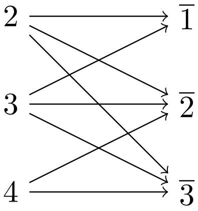


$L(3412)$

Lemma (Donten-Bury–Escobar–Portakal, '23)

The dimension of the weight cone of the $T \times T$-action on Y_w is

$$\dim(\sigma_w) = |V(G^w)| - |\mathcal{C}(G^w)| = \dim(T \times T).$$

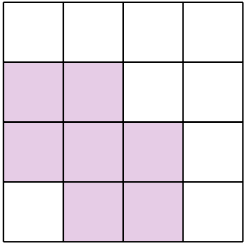


G^{3412}

Computing the complexity of Y_w

The complexity of Y_w is given by

$$
\begin{aligned}
d &= \dim(Y_w) - \dim(T \times T) \\
&= |L'(w)| - \dim(\sigma_w) \\
&= |L'(w)| - |V(G^w)| + |\mathcal{C}(G^w)|.
\end{aligned}
$$

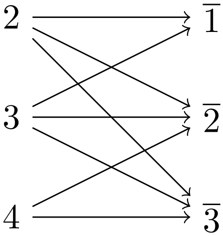


$L'(3412)$

Example ($\overline{X_{3412}} = Y_{3412} \times \mathbb{C}^7$)

Y_{3412} is a $T \times T$-variety of complexity-2 since

$$
\begin{aligned}
d &= |L'(3412)| - \left|V(G^{3412})\right| + |\mathcal{C}(G^w)| \\
&= 7 - 6 + 1 \\
&= 2
\end{aligned}
$$



G^{3412}

Which Y_w are toric?

Theorem (Escobar–Mészáros, '16)

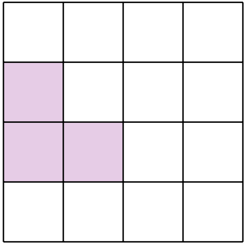
Y_w is toric if and only if $L'(w)$ consists of disjoint hooks not sharing a row or column.

Theorem (Stelzer, '23)

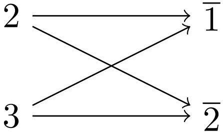
Y_w is toric if and only if w avoids the patterns 1243 and 2143.

Example

Y_{3142} is a toric variety



$L'(3142)$



G^{3142}

How complex can Y_w be?

Theorem (Donten-Bury–Escobar–Portakal, '23)

- *There are no complexity-*1 $T \times T$*-varieties* Y_w
- *There exist* $T \times T$*-varieties* Y_w *of complexity-*d *for* $d \geq 2$.

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- *There are no complexity-*1 *$T \times T$-varieties Y_w*
- *There exist $T \times T$-varieties Y_w of complexity-d for $d \geq 2$.*

Open Problem: Given $d \geq 2$, classify the Y_w of complexity-d.

Maximum complexity of Y_w

Theorem (Escobar–M. '25)

Fix $n \geq 4$ and let $w \in S_n$. The maximum complexity of Y_w is $(n-1)(n-3)$ and is uniquely achieved by the permutation $w = [n, n-1, n-2, \ldots, 3, 1, 2]$.

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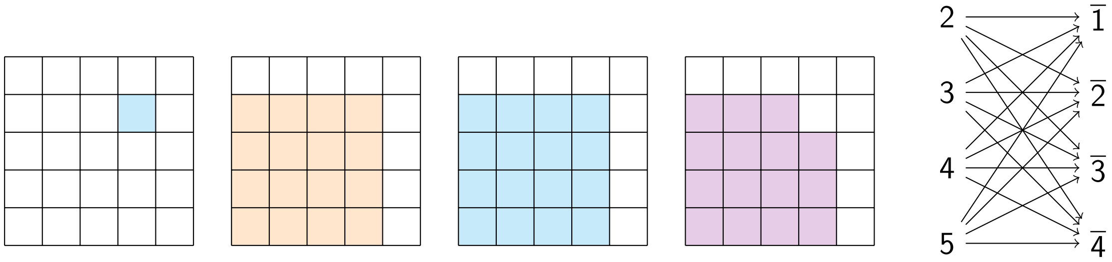

$D^\circ(54312)$ SW(54312) $L(54312)$ $L'(54312)$ G^{54312}

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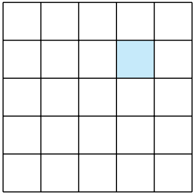
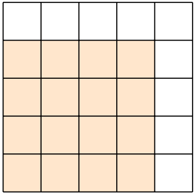
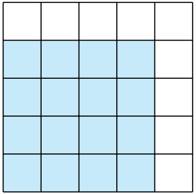
Proof Sketch: $d = |L(w)| + \underbrace{|\mathrm{dom}(w)| - |D^\circ(w)|}_{\leq 0} - |V(G^w)| + |\mathcal{C}(G^w)|.$



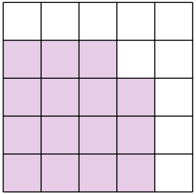
$D^\circ(54312)$



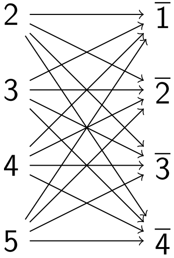
$\mathrm{SW}(54312)$



$L(54312)$



$L'(54312)$

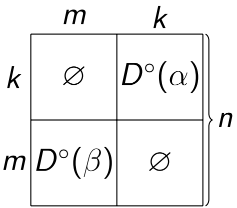


G^{54312}

Lemma

Lemma (Escobar–M. '25)

Let $\alpha \in S_n$ with associated Y_α of complexity d_α such that $D^\circ(\alpha)$ is nonempty and contained in the northeasternmost $k \times k$ submatrix. Let $m = n - k$ and let $\beta \in S_n$ such that $D^\circ(\beta)$ is contained in the southwesternmost $m \times m$ matrix. Then, Y_w, where $w = [\beta_1, \ldots, \beta_m, \alpha_{m+1}, \ldots, \alpha_n]$, has complexity $d_\alpha - |D^\circ(\beta)|$.



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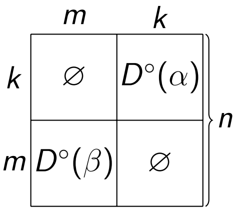
Proof:

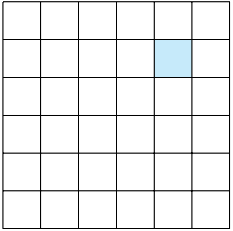
$$\begin{aligned} d_w &= |\mathrm{SW}(w)| - |D^\circ(w)| - |V(G^w)| + |\mathcal{C}(G^w)| \\ &= |\mathrm{SW}(\alpha)| - (|D^\circ(\alpha)| + |D^\circ(\beta)|) - |V(G^\alpha)| + |\mathcal{C}(G^\alpha)| \\ &= d_\alpha - |D^\circ(\beta)| \end{aligned}$$



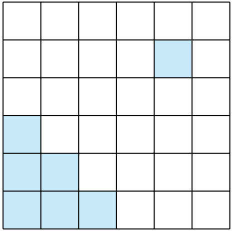
Main Result

Theorem (Escobar–M. '25)

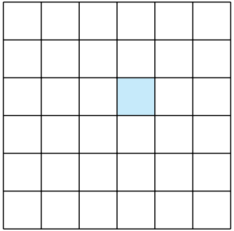
Fix $n \geq 4$ *and let* $w \in S_n$*. There exists* $T \times T$*-varieties* Y_w *of complexity-*d *for* $d \in \{0, 2, 3, \ldots, (n-1)(n-3)\}$.



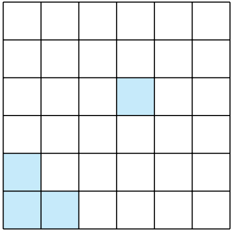
$D^\circ(654312)$, $d = 15$



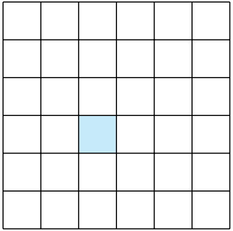
$D^\circ(345612)$, $d = 9$



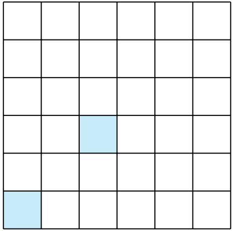
$D^\circ(654231)$, $d = 8$



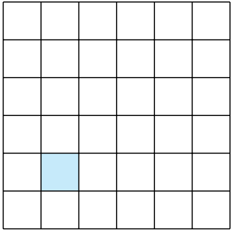
$D^\circ(456231)$, $d = 5$



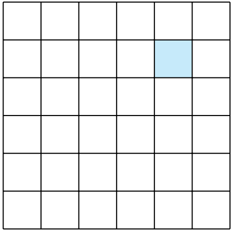
$D^\circ(653421)$, $d = 3$



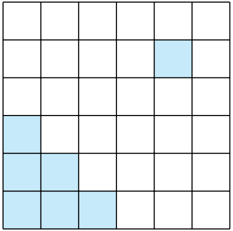
$D^\circ(563421)$, $d = 2$



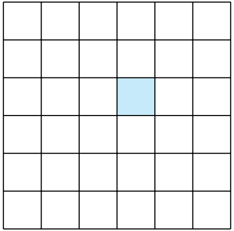
$D^\circ(645321)$, $d = 0$



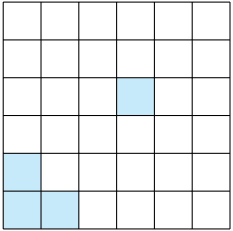
$D^\circ(654312)$, $d = 15$



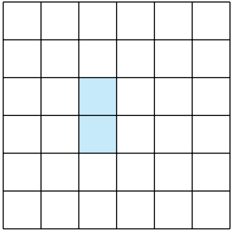
$D^\circ(345612)$, $d = 9$



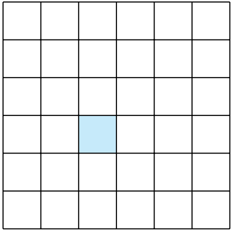
$D^\circ(654231)$, $d = 8$



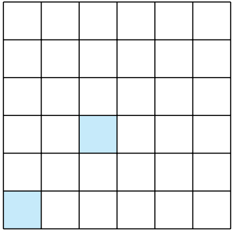
$D^\circ(456231)$, $d = 5$



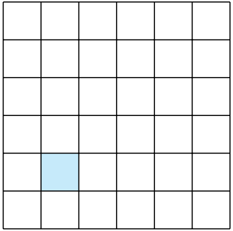
$D^\circ(652431)$, $d = 4$



$D^\circ(653421)$, $d = 3$



$D^\circ(563421)$, $d = 2$



$D^\circ(645321)$, $d = 0$

THANK YOU!