Tableau Formula for Vexillary Double Edelman–Greene Coefficients

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Schubert polynomials

- Complete flag variety

$$\mathcal{F}\ell(\mathbb{C}^n) = \{\ \{0\} \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n : \dim(V_k) = k\ \}$$

decomposes into *Schubert varieties* indexed by $w \in S_n$

- Borel (1953) showed

$$H^*(\mathcal{F}\ell(\mathbb{C}^n);\mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_n]/\langle \text{nonconstant sym. poly.} \rangle$$

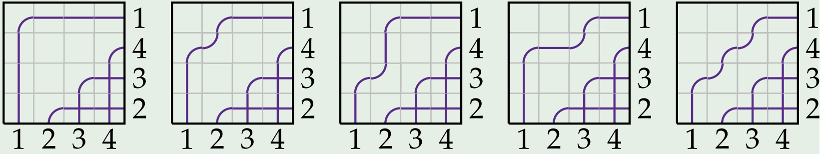
Theorem (Lascoux–Schützenberger, 1982)

Identified Schubert polynomials $\mathfrak{S}_w$ as reps. for these cohom. classes

Bumpless Pipe Dreams (BPDs)

Combinatorial model to compute Schubert polynomials

Example



$$\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$$

Stanley symmetric function

Stanley defined a symmetric function

$$F_w(\mathbf{x}) = \lim_{k\to\infty} \mathfrak{S}_{1^k \oplus w}(\mathbf{x})$$

as a stable limit of Schubert polynomials

Theorem (Stanley, 1984)

$F_w(\mathbf{x})$ is a Schur function $\iff$ w is vexillary

Edelman–Greene coefficients

The a^w_λ below are called *Edelman–Greene coefficients*:

$$F_w(\mathbf{x}) = \sum_\lambda a^w_\lambda \cdot s_\lambda(\mathbf{x}) \quad \text{with} \quad a^w_\lambda \in \mathbb{Z}$$

Theorem (Edelman–Greene, 1987)

Each a^w_λ is actually non-negative and counts something (e.g., tableaux)

Back-stable Schubert polynomials

Lam–Lee–Shimozono, 2018

Studied an *infinite* flag variety (see Section 6 of [LLS18])

- set of all "admissible" flags
- ind-finite variety over $\mathbb{C}$
- decomposes into Schubert varieties

Theorem (Theorem 6.7 of LLS18)

Back-stable Schubert's $\overleftarrow{\mathfrak{S}}_w$ are reps. for these cohom. classes, where

$$w \text{ is an } S_{\mathbb{Z}} \text{ permutation}$$

$\mathbb{Z}$-Permutations

$$S_{\mathbb{Z}} = \{\text{perm. of } \mathbb{Z} \text{ fixing all but finitely many values}\}$$
$$= \left\langle \sigma_k \text{ for } k \in \mathbb{Z} : \begin{array}{ll} \sigma_i^2 = 1 & \text{for } i \in \mathbb{Z}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i \in \mathbb{Z} \end{array} \right\rangle$$

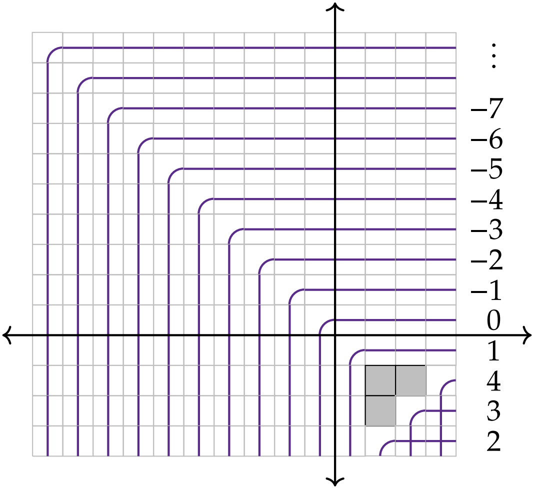
Can use one-line notation with a bar " | ":

Example

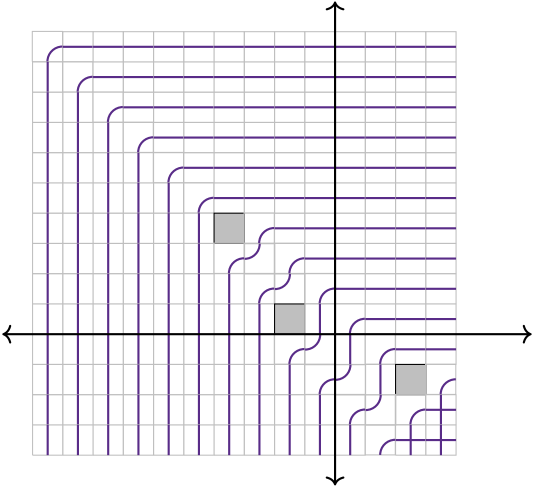
Observe $w = \sigma_2\ \sigma_0\ \sigma_{\overline{1}}\ \sigma_1 = \ldots\ \overline{3}\ \overline{2}\ 1\ \overline{1}\ |\ 3\ 0\ 2\ 4\ 5 \ldots$

Omit the bar if w in S_n (i.e., all non-positive values fixed)

$\mathbb{Z}$-BPDs



$\mathbb{Z}$-BPDs



Back-stable double Schubert

Definition (Lam–Lee–Shimozono, 2018)

The *back-stable double Schubert* for $w \in S_{\mathbb{Z}}$ is

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x};\mathbf{y}) = \sum_{B \in \mathbb{Z}\text{-BPD}(w)} \mathrm{wt}(B) \quad \text{where} \quad \mathrm{wt}(B) = \prod_{(i,j) \text{ is a } \square} (x_i - y_j)$$

Say $f(\mathbf{x})$ *back-stable* if there is some n with $\sigma_i \cdot f = f$ for $i \leq n$

Double Schur

w is called k-Grassmannian if $\{i : w(i) > w(i{+}1)\} = \{k\}$

partitions are in bijection with 0-Grass. :

$$\lambda \mapsto w_\lambda \quad \text{where} \quad w_\lambda(i) = \begin{cases} i - \lambda'_i & i > 0 \\ i + \lambda_{1-i} & i \leq 0 \end{cases}$$

double Schur may be defined $s_\lambda(\mathbf{x}_-; \mathbf{y}) = \overleftarrow{\mathfrak{S}}_{w_\lambda}(\mathbf{x}_-; \mathbf{y})$

Have

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x}; \mathbf{y}) = \sum_\lambda b_\lambda^w(\mathbf{x}; \mathbf{y}) \cdot s_\lambda(\mathbf{x}_-; \mathbf{y})$$

where each $b_\lambda^w(\mathbf{x}; \mathbf{y})$ is a polynomial

Double Stanley symmetric functions

Have

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x};\mathbf{y}) = \sum_\lambda b^w_\lambda(\mathbf{x};\mathbf{y}) \cdot s_\lambda(\mathbf{x}_-;\mathbf{y})$$

where each $b^w_\lambda(\mathbf{x};\mathbf{y})$ is a polynomial

Definition (Lam–Lee–Shimozono, 2018)

$$F_w(\mathbf{x};\mathbf{y}) = \sum_\lambda a^w_\lambda(\mathbf{y}) \cdot s_\lambda(\mathbf{x}_-;\mathbf{y}) \quad \text{where} \quad a^w_\lambda(\mathbf{y}) := b^w_\lambda(\mathbf{x};\mathbf{y})|_{x_i \mapsto y_i}$$

The $a^w_\lambda(\mathbf{y})$ are called *double Edelman–Greene* coefficients

Theorem (Lam–Lee–Shimozono, 2018)

$F_w(\mathbf{x};\mathbf{y})$ *is a double Schur* $\iff$ w *is* 0*-Grassmannian*

Graham positivity

Let $\prec$ be the order on $\mathbb{Z}$ with $1 \prec 2 \prec \ldots \prec \overline{2} \prec \overline{1} \prec 0$

Say $f(\mathbf{y})$ is *Graham positive* if $f \in \mathbb{Z}_{\geq 0}[y_i - y_j \mid i \prec j]$

Theorem (LLS18; Anderson, 2023)

Every $a_\lambda^w(\mathbf{y})$ is Graham positive, for geometric reasons

Example

$$\begin{aligned} a_{(2,2)}^{34512}(\mathbf{y}) &= y_3^2 - y_3 y_0 - y_{\overline{1}} y_3 + y_{\overline{1}} y_0 \\ &= (y_3 - y_{\overline{1}})(y_3 - y_0) \end{aligned}$$

More examples

Example

$$a^{14532}_{(2,2)}(\mathbf{y}) = y_3 + y_4 - y_{\bar{1}} - y_0 = (y_3 - y_{\bar{1}}) + (y_4 - y_0)$$

Example

$$\begin{aligned} a^{345162}_{(2,2)}(\mathbf{y}) = & - y_{\bar{1}}^2 y_0 - y_{\bar{1}} y_0^2 + y_{\bar{1}}^2 y_3 + 2 y_{\bar{1}} y_0 y_3 \\ & + y_0^2 y_3 - y_{\bar{1}} y_3^2 - y_0 y_3^2 + y_{\bar{1}} y_0 y_4 \\ & - y_{\bar{1}} y_3 y_4 - y_0 y_3 y_4 + y_3^2 y_4 + y_{\bar{1}} y_0 y_5 \\ & - y_{\bar{1}} y_3 y_5 - y_0 y_3 y_5 + y_3^2 y_5 \end{aligned}$$

Main result

Theorem (G.–Hamaker–Yu 2024)

Combinatorial proof that **vexillary** $a_\lambda^w(\mathbf{y})$ *are Graham positive*

Say w is vexillary if it avoids the pattern 2143, i.e., has no $i < j < k < \ell$ for which $w(j) < w(i) < w(\ell) < w(k)$

Example

$$a_{(2,2)}^{14532}(\mathbf{y}) = y_3 + y_4 - y_{\overline{1}} - y_0 = (y_3 - y_{\overline{1}}) + (y_4 - y_0)$$

Vexillary permutations

Theorem (Wachs, 1985)

Each vexillary $w \in S_{\mathbb{Z}}$ determines a shape $\lambda(w)$ and flag $\phi(w)$

Example

For $w = 345162$: $\lambda(w) = (2,2,2,1)$ and $\phi(w) = (1,2,3,5)$

Corollary (see also Weigandt, 2021)

For $w \in S_{\mathbb{Z}}$ vexillary one has

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x};\mathbf{y}) = s^{\phi(w)}_{\lambda(w)}(\mathbf{x};\mathbf{y})$$

$\mathbb{Z}$-tableaux

$$T = \begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{1} & 1 & 1 \\ \hline \bar{2} & 0 & 2 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array}$$

$$\begin{aligned} \mathrm{wt}(T) &= (x_{\bar{3}} - y_{\bar{3}+1-1})(x_{\bar{1}} - y_{\bar{1}+2-1}) \dots (x_3 - y_{3+1-3}) \\ &= (x_{\bar{3}} - y_{\bar{3}})(x_{\bar{1}} - y_0) \dots (x_2 - y_3)(x_3 - y_1) \end{aligned}$$

Flag ϕ says max entry in row i is ϕ_i, so

$$s_\lambda(\mathbf{x}_-;\mathbf{y}) = s_\lambda^{(0,\dots,0)}(\mathbf{x}_-;\mathbf{y}) = \sum_T \mathrm{wt}(T)$$

summed over $\mathrm{SSYT}^{(0,\dots,0)}(\lambda)$

Methods

Weigandt $\mathrm{BPD}(v) \leftrightarrow \mathrm{SSYT}^{\phi}(\lambda)$ extends to $\mathbb{Z}$- setting:

$$\overleftarrow{\mathfrak{S}}_v(\mathbf{x};\mathbf{y}) = \sum_B \mathrm{wt}(B) = \sum_T \mathrm{wt}(T)$$

Sample T looks something like

$$T = \begin{array}{|c|c|c|c|} \hline \overline{3} & \overline{1} & 1 & 1 \\ \hline \overline{2} & 0 & 2 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array}$$

Positive flags

For vexillary v whose flag ϕ has only positive entries, each

$T \in \mathrm{SSYT}^{\phi}(\lambda)$ decomposes into (T_-, T_+):

$$T = \begin{array}{|c|c|c|c|} \hline \bar{3} & \bar{1} & 1 & 1 \\ \hline \bar{2} & 0 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$\mathrm{SSYT}^{\phi}(\lambda) = \bigsqcup_{\mu \subseteq \lambda} \left(\mathrm{SSYT}^{(0,\ldots,0)}(\mu) \times \mathrm{SSYT}_{+}^{\phi}(\lambda/\mu) \right)$$

Positive flags

On a functional level:

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x};\mathbf{y}) = s_\lambda^\phi(\mathbf{x};\mathbf{y}) = \sum_{\mu\subseteq\lambda} s_{\lambda/\mu}^\phi(\mathbf{x}_+;\mathbf{y})\cdot s_\mu(\mathbf{x}_-;\mathbf{y})$$

Hence

$$a_\mu^w(\mathbf{y}) = s_{\lambda/\mu}^\phi(\mathbf{x}_+;\mathbf{y})|_{\mathbf{x}\mapsto\mathbf{y}} = \sum_{T\in\mathrm{SSYT}_+^\phi(\lambda/\mu)} \prod_{(r,c)} (y_{T(r,c)} - y_{T(r,c)+c-r})$$

Above main diagonal ($c > r$) already Graham positive

Positive flags

$$T = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \qquad (y_1 - y_3)(y_1 - y_4)(y_2 - y_3)(y_3 - y_1)$$

We solve this by exploiting symmetries of $s^{\phi}_{\lambda/\mu}(\mathbf{x}_+;\mathbf{y})$

Lemma

If i does not appear in ϕ then

$$s^{\phi}_{\lambda/\mu}(\sigma_i \cdot \mathbf{x}_+;\mathbf{y}) = s^{\phi}_{\lambda/\mu}(\mathbf{x}_+;\mathbf{y})$$

Non-positive flags

For instance, with $\lambda = (3, 2)$ and $\phi = (\bar{1}, 0)$

$$
\begin{array}{|c|c|c|}\hline \bar{1} & \bar{1} & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}
-
\left(
\begin{array}{|c|c|c|}\hline \bar{1} & \bar{1} & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}
+
\begin{array}{|c|c|c|}\hline \bar{1} & \bar{1} & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}
\right)
+
\begin{array}{|c|c|c|}\hline \bar{1} & \bar{1} & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array}
$$

$\mu = (3, 2)$ $\mu = (3, 1)$ $\mu = (2, 2)$ $\mu = (2, 1)$

We found a way around this mess by shifting variables

More results

Theorem (G.–Hamaker–Yu 2024)

Our methods work for vex. in the non-reduced setting

Theorem (G.–Hamaker–Yu 2024)

For vex., our proof further refines what Anderson (2023) showed

K-theoretical extension

Split set-valued tableau

$$T = \begin{array}{|c|c|c|}\hline \bar{2},\bar{1} & \bar{1},1 & 1,3 \\ \hline 0,1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

into

$$T' = \begin{array}{|c|c|c|}\hline \bar{2},\bar{1} & \bar{1} & \\ \hline 0 & & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad \text{and} \quad T'' = \begin{array}{|c|c|c|}\hline & 1 & 1,3 \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Future Directions

Open problem

Provide combinatorial proof of positivity for any $w \in S_{\mathbb{Z}}$

Thank you for listening!