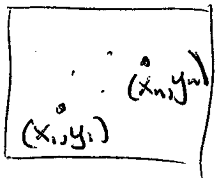
Hilbert Schemes and Standard Monomial Theory

$\mathrm{Hilb}^n \mathbb{C}^2 = \left\{ I \subseteq \mathbb{C}[x,y] \;\middle|\; \dim \mathbb{C}[x,y]/I = n \right\}$ is a smooth, irred. variety



$\rightsquigarrow \quad I = \bigcap_i (x - x_i, y - y_i)$

"collisions" are allowed: eg $\langle x, y^2 \rangle \in \mathrm{Hilb}^2 \mathbb{C}^2$.

Polynomials in $x_1, y_1, \dots, x_n, y_n$ symmetric w.r.t permutation of the n points give global functions on $\mathrm{Hilb}^n \mathbb{C}^2$

$A = \{$alternating polynomials in $x_1, y_1, \dots, x_n, y_n\}$

("Plücker coordinates")

Theorem (Haiman)

The Homogeneous coordinate ring of $\mathrm{Hilb}^n \mathbb{C}^2$ is generated in degree 1 by A.

$$R = \underbrace{A^0}_{\text{symmetric}} \oplus A \oplus A^2 \oplus A^3 \oplus \cdots \qquad (\text{"Plücker algebra"})$$

Theorem (C. '23)

A^d has a $\mathbb{C}$-linear basis indexed by the integer points in a dilation of a certain polyhedron $d \cdot \Delta \subseteq \mathbb{R}^{2n}$

(Δ = N.O. body of $\mathrm{Hilb}^n \mathbb{C}^2$)

(Integer points are leading term exponents)

There are relations between products, this gives you a set of "standard monomials"

Theorem (C.'23)

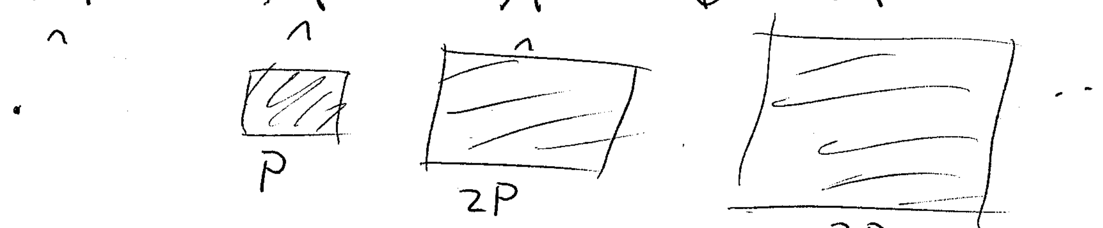
X = smooth toric surface $\leftrightsquigarrow$ P polygon

the homogeneous coordinate ring of $\mathrm{Hilb}^n X$ is the ~~restriction~~ subset of $R = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$

with $f \in A^d$ supported on $d \cdot P$ in each pair of variables (x_i, y_i).

E.g. $\mathbb{P}^1 \times \mathbb{P}^1 \leftrightsquigarrow$ 

$$A^0 \oplus A^1 \oplus A^2 \oplus A^3 \cdots$$



Open problem when is this ring O?

Theorem (C. '24)

$\mathbb{C}$-linear basis for graded pieces of coord. ring of $\mathrm{Hilb}^n(\mathbb{P}^1 \times \mathbb{P}^1)$ (indexed by $\mathbb{Z}$ points in a polytope) with respect to any closed embedding (ample case)

Proof is non-constructive! Show leading term exps are in the set, and show that the count is correct using $\mathbb{C}^2$ case + geometry.