$GL(n)$-orbits on Two Complete Flag Varieties and a Line

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I: Intro

Let $G = GL(n, \mathbb{C})$

$\mathcal{B} =$ flag variety of G.

Of course $\mathcal{B} = G/B_+$, where $B_+ =$ std upper Δ Borel subgroup.

Consider: G-diagonal orbits on the triple product $\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}$.

These orbits have been studied extensively by Magyar, Travkin, and others.

They are related to study of mirabolic $\mathcal{D}$-modules which play an important role in study of category $\mathcal{O}$ for rational Cherednik algebras.

Goal: Describe closure ordering and geometry of $G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1})$ using a variant of the product of Bruhat orders on $\mathcal{S}_n \times \mathcal{S}_n$.

Approach: Suffices to understand geometry of a certain subset of $G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1})$.

This subset is given by action of "little Borel" on $\mathcal{B}$.

Orbits of "Little Borel":

Embed $G_{n-1} := GL(n-1) \subset G$ in the upper left corner.

$B_{n-1} =$ std upper Δ Borel of G_{n-1} embedded in B_+ in upper left corner.

FACT: B_{n-1} acts on $\mathcal{B}$ with finitely many orbits.

Let B^* be the Borel subgroup that stabilizes the flag:

$$\mathcal{E}^* := (\mathcal{E}_1^* \subset \ldots \subset \mathcal{E}_i^* \subset \ldots \subset \mathcal{E}_n^*),$$

$$\mathcal{E}_i^* = \text{span}\{e_n, e_1, \ldots, e_{i-1}\}.$$

(Here $e_j = j$-th standard basis vector of $\mathbb{C}^n$.)

NOTE: $B_{n-1}Z = B_+ \cap B^*$ with $Z =$ centre of G.

Theorem: The B_{n-1}-orbits on $\mathcal{B}$ are precisely the non-empty intersections of B and B^*-orbits on $\mathcal{B}$.

Bruhat decomposition $\Rightarrow$ For $Q \in B_{n-1}\backslash\mathcal{B}$,

$Q = (BwB/B) \cap (B^*u^*B^*/B^*)$.

Def'n: For $Q \in B_{n-1}\backslash\mathcal{B}$, the *Shareshian pair* (or $\mathcal{S}h$-*pair*) associated to the orbit Q is

$\mathcal{S}h(Q) = (w, u^*) \in \mathcal{S}_n \times \mathcal{S}_n \Leftrightarrow$

$Q = (BwB/B) \cap (B^*u^*B^*/B^*)$.

Theorem: The closure relations on $B_{n-1}\backslash\mathcal{B}$ can be described by $\mathcal{S}h$-Bruhat ordering:

$$\overline{Q} = \overline{(BwB/B)} \cap \overline{(B^*u^*B^*/B^*)}.$$

i.e. $\overline{Q}$ is the *intersection* of two Schubert varieties (albeit) with respect to different Borel subgroups.

Further, the "extended" Richardson-Springer monoid action on $B_{n-1}\backslash\mathcal{B}$ can be understood using a version of the classical monoid action of $\mathcal{S}_n$ on itself extended diagonally to the product $\mathcal{S}_n \times \mathcal{S}_n$.

Back to "Big Picture": $G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1})$.

FACT: $B_{n-1}\backslash\mathcal{B}$ embeds in $G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1})$ as follows:

First Observe:

$$G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}) \longleftrightarrow B_+\backslash(\mathcal{B} \times \mathbb{P}^{n-1}).$$

B_+-orbits on $\mathcal{B} \times \mathbb{P}^{n-1}$ are determined by projection to second factor:

Let $\mathcal{O}_i = B_+ \cdot [e_i] \subset \mathbb{P}^{n-1}$.

Then

$$B_+ \backslash (\mathcal{B} \times \mathbb{P}^{n-1}) = \coprod_{i=1}^{n} B_+ \backslash (\mathcal{B} \times \mathcal{O}_i).$$

We can reduce things one more time:

Let $S_i := \mathsf{Stab}_{B_+}([e_i]) \subset B_+$

$$B_+\backslash(\mathcal{B} \times \mathcal{O}_i) \longleftrightarrow S_i\backslash\mathcal{B}.$$

NOTE: For $i = n$, $S_n = B_{n-1}Z$ with $Z =$ centre of G, i.e.

$B_+\backslash(\mathcal{B} \times \mathcal{O}_n) \longleftrightarrow B_{n-1}\backslash\mathcal{B}.$

To understand $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1})$ suffices to understand a subset of "Little Borel" orbits in higher rank.

Let $G_{n+1} = GL(n+1)$:

$G \subset G_{n+1}$ in top left hand corner.

$\mathcal{B}_{n+1} =$ (flag variety of G_{n+1}) $\cong G_{n+1}/B_{+,n+1}$

and embed $B_+ \subset B_{+,n+1}$ in the top left corner as before:

MAIN THEOREM: There is a Zariski open, B_+-stable subvariety $\mathfrak{X}$ of $\mathcal{B}_{n+1}$

such that there exists a 1-1 correspondence

$$G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}) \longleftrightarrow B_+\backslash\mathfrak{X}$$

which preservers the closure ordering and intertwines Richardson-Springer monoid actions on either set of orbits (with small caveats).

Consequences:

Clearly, $B_+\backslash\mathfrak{X} \subset B_+\backslash\mathcal{B}_{n+1}$.

Upshot:

(1) Closure ordering on $B\backslash\mathfrak{X}$ and therefore on $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1})$ can be described by $\mathcal{S}h$-ordering which is a variant on product of Bruhat order on $\mathcal{S}_{n+1} \times \mathcal{S}_{n+1}$.

(2) Richardson-Springer monoid action on $G\backslash(\mathcal{B}\times \mathcal{B} \times \mathbb{P}^{n-1})$ can be understood in terms of classical monoid action of $\mathcal{S}_{n+1}$ on itself extended to product $\mathcal{S}_{n+1} \times \mathcal{S}_{n+1}$ diagonally.

Further, since $B_+\backslash\mathcal{B}_{n+1} \subset G_{n+1}\backslash(\mathcal{B}_{n+1}\times\mathcal{B}_{n+1}\times\mathbb{P}^n)$.

We also obtain an embedding of sets of orbits

$$G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1}) \hookrightarrow G_{n+1}\backslash(\mathcal{B}_{n+1}\times\mathcal{B}_{n+1}\times\mathbb{P}^n).$$

where the embedding respects the closure ordering.

Comparing with Magyar:

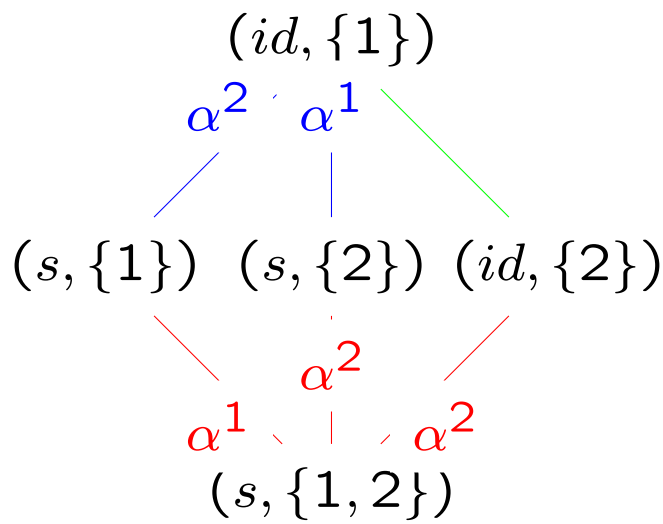
Magyar parameterizes $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1})$ by so-called decorated permutations:

Decorated Permutation: (w,Δ), where $w \in \mathcal{S}_n$ and $\Delta = \{j_1 < \ldots < j_k\} \subset \{1,\ldots,n\}$ is a *descending* sequence for w^{-1}.

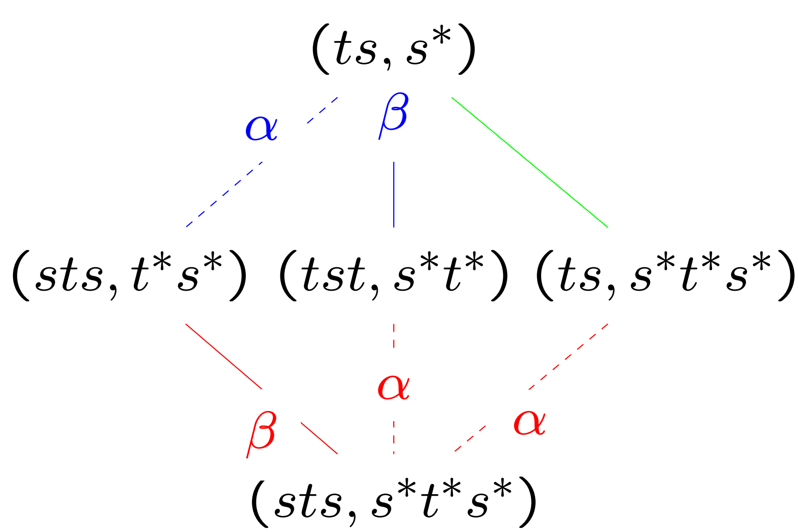
i.e. $\{w^{-1}(j_k) < \ldots < w^{-1}(j_1)\}$.

Magyar describes closure ordering on $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1})$ using subtle combinatorial ordering on set of all decorated permutations. No description of monoid action is given.

Example of $n = 2$: $GL(2)\backslash(\mathcal{B}_2 \times \mathcal{B}_2 \times \mathbb{P}^1)$:



$B_2\backslash\mathfrak{X} \subset B_2\backslash\mathcal{B}_3$: $s = (1, 2)$, $t = (2, 3)$; $s^* = (1, 3)$, $t^* = (1, 2)$



Sketch of Proof of Main Theorem:

The correspondence $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1}) \longleftrightarrow B_+\backslash\mathfrak{X}$ is constructed locally.

Recall:

$G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1}) \longleftrightarrow B_+\backslash(\mathcal{B}\times\mathbb{P}^{n-1})$.

For $\mathcal{O}_i = B_+ \cdot [e_i] \subset \mathbb{P}^{n-1}$,

$B_+\backslash(\mathcal{B}\times\mathbb{P}^{n-1}) \longleftrightarrow \coprod_{i=1}^{n} B_+\backslash(\mathcal{B}\times\mathcal{O}_i)$.

Let $S_i = \mathrm{Stab}_{B_+}[e_i]$, then

$B_+\backslash(\mathcal{B}\times\mathcal{O}_i) \longleftrightarrow S_i\backslash\mathcal{B}_n$.

STEP 1:

For each $i = 1, \ldots, n$, develop a theory of i-$\mathcal{S}h$-pairs for S_i-orbits on $\mathcal{B}$.

FACT 1: $\exists$ a Borel subgroup $B^i \subset G$ such that the $S_i - orbits$ on $\mathcal{B}$ are precisely the non-empty intersections of B and B^i-orbits on $\mathcal{B}$.

$\Rightarrow$ For $Q \in S_i \backslash \mathcal{B}$, we can define:

$\mathcal{S}h_i(Q) := (w, u^i) \in \mathcal{S}_n \times \mathcal{S}_n \Leftrightarrow$

$Q = (BwB/B) \cap (B^i u^i B^i/B^i)$.

$\Rightarrow$ Can describe closure ordering, monoid actions, etc using product of Bruhat orders on $\mathcal{S}h_i$-pairs just as for B_{n-1}-orbits.

STEP 2:

$G = GL(n)$ acts on flag variety $\mathcal{B}_{n+1}$ of G_{n+1} with finitely many orbits.

(Up to centre G is a symmetric subgroup of G_{n+1}.)

These orbits are classified by Yamamoto, Matsuki-Oshima, etc.

FACT 2: For every $i = 1, \ldots, n$, $\exists$ a G-orbit $\mathcal{Q}(i)$ on $\mathcal{B}_{n+1}$ and a 1-1 correspondence, preserving the closure ordering, intertwining monoid actions, etc:

$$S_i \backslash \mathcal{B}_n \longleftrightarrow B_+ \backslash \mathcal{Q}(i) \subset B_+ \backslash \mathcal{B}_{n+1},$$

STEP 3: Piece together correspondences in STEP 2 to prove main result:

$$G\backslash(\mathcal{B} \times \mathcal{B} \times \mathbb{P}^{n-1}) \longleftrightarrow B_+\backslash\mathfrak{X},$$

The Zariski open subvariety $\mathfrak{X} := \coprod_{i=1}^{n} \mathcal{Q}(i)$ is the disjoint union of G-orbits on $\mathcal{B}_{n+1}$ from STEP 2.

The "local" correspondence in STEP 2:

$S_i\backslash\mathcal{B}_n \longleftrightarrow B_+\backslash\mathcal{Q}(i) \subset B_+\backslash\mathcal{B}_{n+1}$

then glues together to give the "global" one on the level of sets of orbits.

HOWEVER: Proving the "global" correspondence preserves closure relations is subtle,

Strategy:

DESCRIBE:

$$G\backslash(\mathcal{B}\times\mathcal{B}\times\mathbb{P}^{n-1}) \longleftrightarrow B\backslash\mathfrak{X} \hookrightarrow G_{n+1}\backslash(\mathcal{B}_{n+1}\times\mathcal{B}_{n+1}\times\mathbb{P}^{n}).$$

in terms of decorated permutations and then show Magyar's ordering on decorated permutations is preserved.

PROBLEM: The "local" correspondence is not easy to describe using decorated permutations.

SOLUTION: HOWEVER we can

DESCRIBE $S_i\backslash\mathcal{B}_n \longleftrightarrow B\backslash\mathcal{Q}(i) \subset B\backslash\mathcal{B}_{n+1}$ in terms of $\mathcal{S}h$-data.

Translating the correspondence in terms of $\mathcal{S}h$-data into decorated perms is relatively straightforward.

Future Goals:

(1) Generalize $\mathcal{S}h$ picture to describe B_+-orbits on the product $\mathcal{B} \times X_w$, where X_w is a Schubert cell in a Grassmannian which is a toric variety.

(2) Applications to Representation Theory and Other Combinatorics:

The theory of B_{n-1}-orbits on Gr(k, n) has a particularly nice combinatorial description both in terms of $\mathcal{S}h$-pairs and other combinatorial data (i.e. painted Young diagrams).

The geometric and combinatorial data line up nicely with the structure of cyclic $U(\mathfrak{b}_{n-1})$-submodules of $\bigwedge^k \mathbb{C}^n$.

(3) What about the case where $G = SO(n)$??