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Parabolic induction for Springer fibers of type C
by Mee Seong Im

$G = GL_n(\mathbb{C}) \rightsquigarrow \mathcal{N} \subseteq \mathfrak{gl}_n$
$\quad = \{x \in \mathfrak{gl}_n : x^k = 0$ for some $k\}$

G-orbits on $\mathcal{N} \overset{\text{Jordan canonical form}}{\longleftrightarrow} \{\lambda = (\lambda_1, \ldots, \lambda_r) : \lambda_i \geq \lambda_{i+1},\ |\lambda| = \sum_{i=1}^{r} \lambda_i = n\}$
$\qquad = P(n)$

$G \cdot x = \mathcal{O}_\lambda \longleftrightarrow \lambda$

$\text{type}(x) = \lambda$

$\text{Irr}(S_n) \longleftrightarrow P(n)$

$S^\lambda \longleftrightarrow \lambda$
(Specht mod)

Closure ordering

$\lambda, \mu \in P(n)$, $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$

$\Longleftrightarrow$ $\mu_1 \leq \lambda_1$, $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, $\vdots$ (dominance order on partitions)

Resolution of singularities

$B \subseteq G$ Borel subgroup

$\mathfrak{g} = \mathfrak{gl}_n = \mathfrak{n}^- \oplus \underbrace{\mathfrak{h} \oplus \mathfrak{n}^+}_{\mathfrak{b} = \mathrm{Lie}(B)}$, root space decomposition

$$\tilde{\mathcal{N}} = G \times_B \mathfrak{n}^+ \xrightarrow{\pi} \mathcal{N}$$

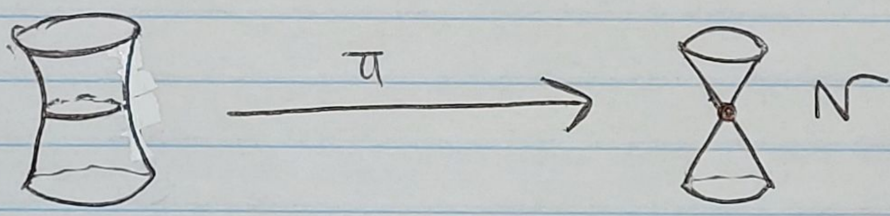
$$(gb^{-1}, bxb^{-1}) \sim (g, x) \longmapsto gxg^{-1}$$

for any $b \in B$

π is called the Springer resolution.

Ex. $GL_2(\mathbb{C}) \curvearrowright \mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \right\}$

Real Pic



Springer fibers Let $x \in \mathcal{O}_\lambda$ (type$(x) = \lambda$).

$$\mathcal{F}_x = \pi^{-1}(x) = \{ \mathfrak{b}' \subseteq \mathfrak{g} : x \in \mathfrak{b}' \}$$
$$= \{ 0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_n = \mathbb{C}^n : \dim(F_i) = i,\ x(F_i) \subseteq F_{i-1}\ \forall i \}$$
$$\subseteq G/B \text{ flag variety}$$

Spaltenstein map $\exists$ surjective map

$$\theta : \mathcal{F}_x \longrightarrow \mathrm{std}(\lambda)$$

that induces a bijection

$$\mathrm{Irr\,Comp}(\mathcal{F}_x) \xleftrightarrow{\sim} \mathrm{std}(\lambda)$$

$$\theta^{-1}(T) \longleftarrow T$$

Cor. $\mathcal{F}_x$ is of pure dimension

$$\dim(\mathcal{F}_x) = \sum_{i=1}^{r} (i-1)\lambda_i .$$

Lusztig - Spaltenstein

$$H^*(\mathcal{F}_x, \mathbb{C}) \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{1}_{S_\lambda})$$

as (graded) S_n-modules.

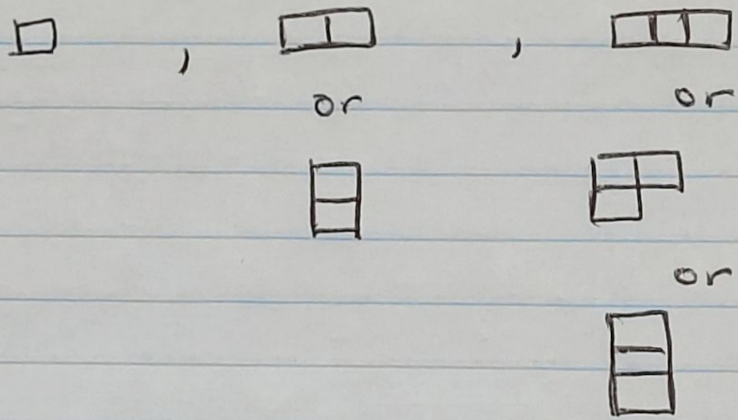
$H^{top}(\mathcal{F}_x, \mathbb{C}) \simeq S^\lambda$ (Specht module)

The map θ in more detail

Let $F_\bullet \in \mathcal{F}_x$, where $F_\bullet = (0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq \mathbb{C}^n)$.

$$\theta(F_\bullet) = (\dots \mathrm{type}(x|_{F_i}) < \mathrm{type}(x|_{F_{i-1}}) < \dots$$
$$\dots < \mathrm{type}(x) = \lambda)$$

Spaltenstein sequences determine the Young tableaux.



<u>Ex.</u> $GL_3(\mathbb{C}) \curvearrowright \mathcal{N}(\mathfrak{gl}_3)$

Get 3 orbits labeled by $(3), (2,1), (1^3)$.

$\mathcal{O}_{(1^3)} \Rightarrow x = 0 \in \mathcal{N} \Rightarrow \mathcal{F}_0 = G/B$

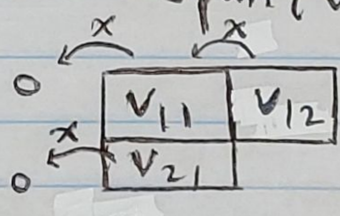
full flag variety

$\mathcal{O}_{(3)} \Rightarrow x \sim \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, $\mathbb{C}^3 = \mathrm{span}\{e_1, e_2, e_3\}$, $0 \xleftarrow{x} e_1 \xleftarrow{x} e_2 \xleftarrow{x} e_3$

$\Rightarrow \mathcal{F}_x \cong pt = \{0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \mathbb{C}^3\}$

$\mathcal{O}_{(2,1)} \Rightarrow x \sim \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right)$, $\mathbb{C}^3 = \mathrm{span}\{e_1, e_2, f_1\}$, $0 \xleftarrow{x} e_1 \xleftarrow{x} e_2$, $0 \xleftarrow{x} f_1$

$= \mathrm{span}\{v_{11}, v_{12}, v_{21}\}$



$\Rightarrow \mathcal{F}_x = C_{\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}} \cup C_{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}} \cong \mathbb{P}^1 \vee \mathbb{P}^1$

since

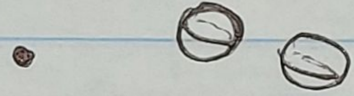
⑤

$$C_{\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}} = \{ 0 \subseteq \langle \alpha v_{11} + \beta v_{21} \rangle \subseteq \langle v_{11}, v_{21} \rangle \subseteq \mathbb{C}^3 : (\alpha, \beta) \in (\mathbb{C}^2)^* \} \cong \mathbb{P}^1$$

$$C_{\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}} = \{ 0 \subseteq \langle v_{11} \rangle \subseteq \langle v_{11}, \gamma v_{12} + \delta v_{21} \rangle \subseteq \mathbb{C}^3 : (\gamma, \delta) \in (\mathbb{C}^2)^* \} \cong \mathbb{P}^1$$

$$C_{\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}} \cap C_{\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}} = \{pt\}.$$

$$H^*(\mathcal{F}_x, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}^2$$



Let $\mathfrak{l} = \mathrm{Lie}(L)$, $L \leq G$, $\mathfrak{l} \subseteq \mathfrak{g}$
Levi subalg.

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ Levi decomposition

Let $\mathcal{O}_0$ be a nilpotent orbit for L.

Lusztig-Spaltenstein $\exists!$ nilpotent orbit $\mathcal{O}$ for G s.t.
$\mathcal{O} \cap (\mathcal{O}_0 + \mathfrak{n})$ is dense in $\mathcal{O}$.

$\Rightarrow \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0) = \mathcal{O}$

Get induced nilpotent orbit!

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{l} = \prod_{i=1}^{r} \mathfrak{gl}_{n_i}$,

where $\sum_{i=1}^{r} n_i = n$,

let $e_0 \in \mathfrak{g}_0$ be nilpotent of type $(\mu_1, \ldots, \mu_r)$ where $\mu_i \vdash n_i$.

Write $\underline{\mu} = (\mu_1, \ldots, \mu_r)$ as multi-partition.

Have Spaltenstein map

$$(*) \quad \begin{array}{ccc} \mathcal{F}_{e_0} & \xrightarrow[\text{Spaltenstein}]{\text{Lusztig-}} & \mathcal{F}_e \\ \prod_{i=1}^{r}\theta_i \downarrow & & \downarrow \theta \\ \prod_{i=1}^{r} \operatorname{std}(\mu_i) & \xrightarrow{\text{stk}} & \operatorname{std}(\lambda). \end{array} \qquad e = e_0 + x, \quad x \in \mathfrak{n}$$

That is, $\exists$ partition λ of n and the map $\text{stk} : \prod_{i=1}^{r} \operatorname{std}(\mu_i) \longrightarrow \operatorname{std}(\lambda)$ which is defined as follows:

$$\text{stk}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \right)$$

$$= \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 8 & 10 \\ \hline 3 & 5 & 7 & 9 \\ \cline{1-4} 11 \\ \cline{1-1}\end{array}$$

So we can find an $e \in \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0)$

of type λ and define a map

$$\begin{array}{ccc} \mathcal{F}_{e_0} & \xrightarrow{\text{Lusztig-Spaltenstein}} & \mathcal{F}_e, \quad e = e_0 + X \\ & & X \in \eta \\ \mathfrak{b}_{e_0} & \longmapsto & \mathfrak{b}_{e_0} + \eta \end{array}$$

Thm (Saunders - Topley '23)

(✱) commutes.

I. - Saunders - Wilbert

Have a stacking map for type C Springer fibers (in-progress).

$Sp_{2n} \curvearrowright \mathcal{N}(\mathfrak{sp}_{2n}) \longleftrightarrow$ {partitions of $2n$ s.t. odd parts occur with even multiplicity}

$\updownarrow$

Weyl group
$W(C_n) = \mathbb{Z}/2\mathbb{Z} \wr S_n$ $\quad \{(\mu, \nu) : \mu_i \geq \nu_i - 1, \ \nu_i \geq \mu_{i+1} - 1 \ \forall i\}$

$\cap$

$\mathrm{Irr}(W(C_n)) = \{(\mu, \nu) : |\mu| + |\nu| = n\} = \mathcal{Q}_n$

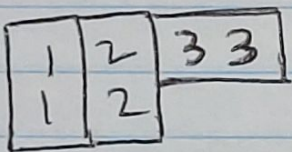
Springer fibers of type C

Let $x \in \mathcal{O}_\lambda$. Then

$$\mathcal{F}_x = \{ 0 \subseteq F_1 \subseteq \dots \subseteq F_{2n-1} \subseteq \mathbb{C}^{2n} : \dim(F_i) = i,\ x(F_i) \subseteq F_{i-1},\ F_i^{\perp} = F_{2n-i} \text{ for all } i \}$$

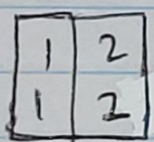
$\mathrm{Irr\,Comp}(\mathcal{F}_x) \longrightarrow ADT(\lambda)$ admissible domino tableaux of shape λ

Ex. If I remove a domino, its Young diagram corresponds to a type C orbit.

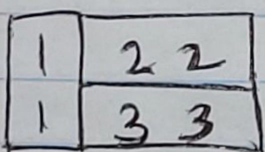


$\lambda = (4, 2)$

$\xrightarrow{\text{remove } 3}$

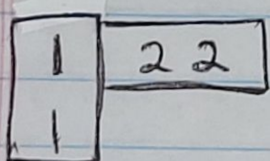


$\lambda' = (2, 2)$



$\lambda = (3, 3)$

$\xrightarrow{\text{remove } 3}$



$\lambda' = (3, 1)$ ← Young diagram doesn't correspond to a type C orbit.

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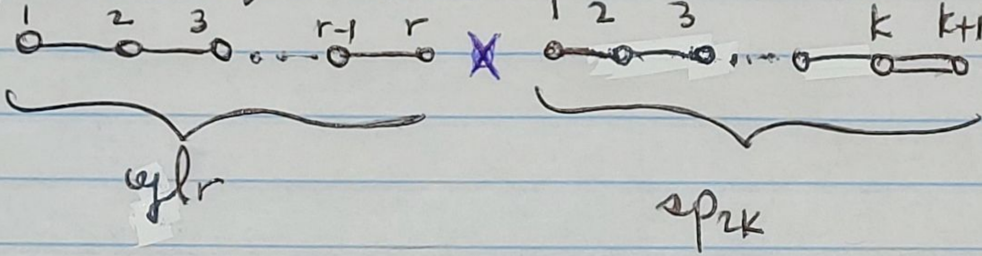
$$\text{Fx} \longrightarrow \text{"signed" ADT}$$

Can restrict to maximal Levi subalgebras in $\mathfrak{sp}_{2n}$:

$$\mathfrak{gl}_r \oplus \mathfrak{sp}_{2k} \subseteq \mathfrak{sp}_{2n}, \quad \text{where } 2r + 2k = 2n.$$

Dynkin diagram:



$$\text{stk}: \ \text{std}(\mu) \times \text{ADT}(\nu) \longrightarrow \text{ADT}(\lambda)$$

$$\text{Ind}^{\mathfrak{sp}_{2n}}_{\mathfrak{gl}_r \oplus \mathfrak{sp}_{2k}} \theta_\mu \times \theta_\nu = \theta_\lambda$$

$$\text{stk}: \ \text{std}(\mu) \times \text{ADT}(1^{2k}) \longrightarrow \text{ADT}(\lambda)$$