On Shelling a Family of Symmetric Spaces

Fernando Díaz Morera
(joint with Aram Bingham)

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Outline

(Strong) Bruhat order: $(\mathfrak{S}_n, <_{\mathrm{BC}})$

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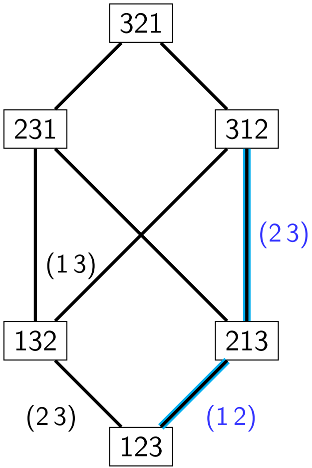

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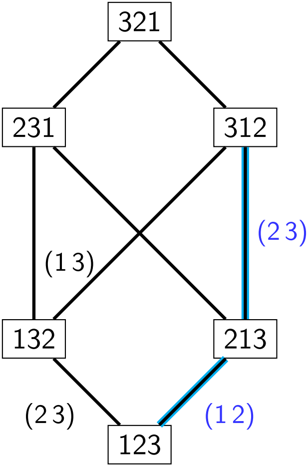

$$\mathbf{T}_n \subset \mathbf{B}_n \subset \mathrm{GL}_n(\mathbb{C}) \rightsquigarrow \mathrm{GL}_n / \mathbf{B} = \bigsqcup \mathbf{B}\, v\, \mathbf{B} / \mathbf{B} \rightsquigarrow X_{v\,\mathbf{B}} := \overline{\mathbf{B}\, v\, \mathbf{B} / \mathbf{B}}.$$
$$\mathfrak{S}_n \cong \mathrm{N}_{\mathrm{GL}_n}(\mathbf{T}) / \mathbf{T}, \quad u <_{\mathrm{BC}} v \iff X_{u\,\mathbf{B}} \subseteq X_{v\,\mathbf{B}}, \quad \ell(v) = \dim X_{v\,\mathbf{B}}$$

Posets

- A poset $\mathcal{P}$ with order relation $<$ is bounded if there are elements $\hat{\mathbf{0}}$ and $\hat{\mathbf{1}}$ such that $\hat{\mathbf{0}} < v < \hat{\mathbf{1}}$ for all v in $\mathcal{P}$.

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- $\mathcal{P}$ is graded if all *maximal* chains have the same length.
- If $\mathcal{P}$ is graded, then the length function $\ell : \mathcal{P} \to \mathbb{N}$ assigns to v in $\mathcal{P}$ the value m where $u_0 \lessdot u_1 \lessdot \cdots \lessdot u_m = v$ for any saturated chain to v.

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Finding the number of partially order sets
with n elements is still unknown...

EL-labeling on $\mathfrak{S}_n$

Let $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_n$ be in $\mathfrak{S}_n$. We say $u \lessdot_{\mathrm{BC}} v$ whether $\ell(v) = \ell(u) + 1$, and

(i) $u_k = v_k$ for k in $\{1, .., \widehat{i}, ..., \widehat{j}, ..., n\}$

(ii) $u_i = v_j, u_j = v_i$, and $u_i < u_j$.

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Let $\Lambda = [n] \times [n]$ denote poset of pair such that $(i, j) \leq (r, s)$ if $i < r$, or $i = r$ and $j < s$.

$$\begin{aligned} C(\mathfrak{S}_n) &\xrightarrow{\quad\quad \eta \quad\quad} \Lambda \\ (u, v) &\longrightarrow \eta(u, v) := (u_i, u_j) \end{aligned}.$$

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E.g. Let $v = 312$ and $e = 123$ be in $\mathfrak{S}_3$. The max chain $e \lessdot_{\mathrm{BC}} 213 \lessdot_{\mathrm{BC}} v$ is the **LEX** smallest. Moreover, $\eta(e, 213) \leq_{\mathbf{LEX}} \eta(213, v)$ is non-decreasing.

Lexicographic shellability

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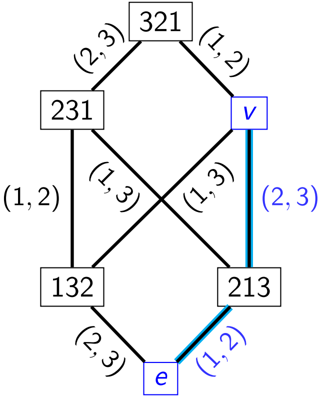
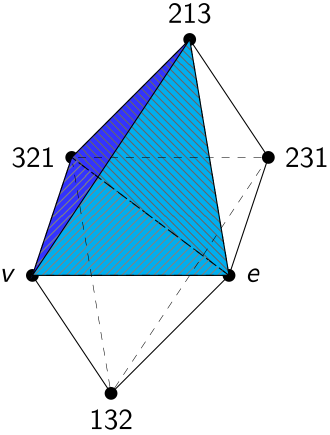
A poset $\mathcal{P}$ is **EL-shellable or lexicographically shellable** if it admits an *EL*-labeling.

EL-labeling of $\mathfrak{S}_3$



$\xrightarrow{\Delta(P)}$



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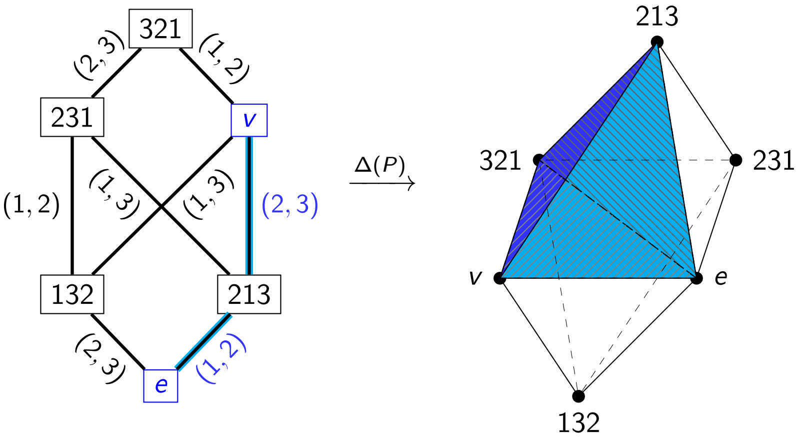

The **order complex** of a poset $\mathcal{P}$ is the abstract *simplicial complex*, denoted $\Delta(\mathcal{P})$, whose k-dimensional faces are the chains $u_0 < u_1 < \cdots < u_k$ of $k+1$ comparable poset elements.

Intermezzo: commutative algebra

A *simplicial complex* Δ is **shellable** if there is an ordering of the maximal faces $F_1, F_2, ..., F_m$ so that for all i, j with $i < j$, there exits $k < j$ such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j \setminus \{p\}$ for some p in F_j.

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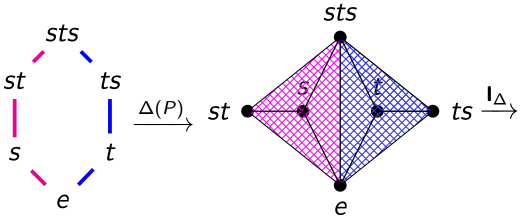

$\mathbf{I}_\Delta = (x_{i_1} \cdots x_{i_r} : \{i_1, \dots, i_r\} \notin \Delta)$

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$S := \mathbf{k}[x_1, x_2, \dots, x_n]$

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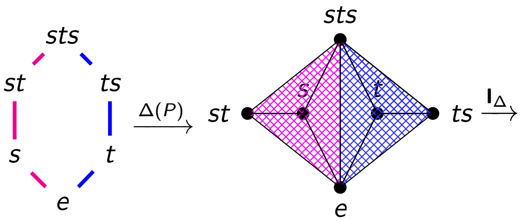

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Theorem (Björner '80)

If $\mathcal{P}$ is a graded poset with an *EL*-labeling, then $\Delta(\mathcal{P})$ is shellable.

Theorem (Kind-Kleinschmidt '79)

The Stanley-Reisner ring of a shellable Δ is *Cohen–Macaulay*.

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- Fixed-point free involutions in $\mathfrak{S}_n$ are EL-shellable (Can-Cherniavsky-Twelbeck, '15).
- The *rook monoid* is EL-shellable (Can, '19).

Symmetric spaces

$$\begin{tikzcd} \mathbf{G} \arrow[d, squiggly] & \mathbf{P} \arrow[l, hook'] \arrow[d, squiggly] & \mathbf{B} \arrow[l, hook'] \arrow[d, squiggly] & \mathbf{T} \arrow[l, hook'] \arrow[d, squiggly] \\ \text{Connected reductive} & \text{Parabolic} & \text{Borel} & \text{Max. Torus} \end{tikzcd}$$

Let $\theta : \mathbf{G} \to \mathbf{G}$ such that $\theta^2 = \mathrm{id}$. We say $\mathbf{G}/\mathbf{K}$ is a symmetric space if $\mathbf{K} = \mathbf{G}^\theta$ (fixed-point subgroup).

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$\mathbf{G} = \mathrm{GL}_n \times \mathrm{GL}_n$, $\theta(g,h) = (h,g)$, $\mathbf{K} = \{(g,g) : g \in \mathrm{GL}_n\}$. So $\mathbf{G}/\mathbf{K} \cong \mathrm{GL}_n$ acted on by $\mathbf{B} = \mathbf{B}_n \times \mathbf{B}_n \subset \mathbf{G}$. In particular, $\mathbf{B}$-orbits are the same as $\mathbf{B}_n\, v\, \mathbf{B}_n$ double cosets, v in $\mathfrak{S}_n$...

Classification (à la Cartan)

Type	$\mathbf{G}$	$\mathbf{K}$	$\mathbf{B} \backslash \mathbf{G} / \mathbf{K}$	Shellable
AI	GL_n(or SL_n)	O_n(or SO_n)	$\mathrm{Invol}(n)$	☑(Incitti, '04)
AII	SL_{2n}	Sp_{2n}	$\mathrm{Invol}^{FPF}(2n)$	☑(Can-C-T, '15)
AIII	GL_{p+q}	$GL_p \times GL_q$	(p, q)-clans	☕✍···
CI	Sp_{2n}	GL_n	(n, n)-clans	🦇
CII	Sp_{p+q}	$Sp_{2p} \times Sp_{2q}$	CII-clans	🦇
BDI	SO_n	$SO_k \times SO_{n-k}$	$(k, n-k)$-clans	🦇
DIII	SO_{2n}	GL_n	DIII-clans	🦇

Clans

Definition (Yamamoto '97 - Wyser '06)

Let p and q be two positive integers such that $p + q = n$. A (p, q)-**clan** is an ordered set of n symbols $c_1 \dots c_n$ such that:

(i) Each symbol c_i is either " $+$ ", " $-$ " or a $\mathbb{N}_{>0}$.

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Example

$\gamma_1 = +1212-$ and $\gamma_2 = +1717-$ are equivalent $(3,3)$-clans. Likewise, $\gamma_3 = 12{+}21$ is a $(3,2)$-clan and $\gamma_4 = +1{+}1$ is a $(3,1)$-clan.

Stemming from...

Theorem (Matsuki-Oshima '90)

Let $\mathcal{C}_{p,q}$ denote the set of (p,q)-clans. Then

$$\mathcal{C}_{p,q} \underset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \mathsf{B}\text{-orbits in } \mathsf{GL}_{p+q} / \mathsf{GL}_p \times \mathsf{GL}_q \\ \text{or} \\ \mathsf{GL}_p \times \mathsf{GL}_q\text{-orbits in } \mathsf{GL}_n / \mathsf{B} \end{matrix} \right\}.$$

Moreover, a clan is matchless if it consists only of $+/-$'s. Hence,

$$\{\text{matchless clans}\} \underset{1:1}{\longleftrightarrow} \{\text{closed orbits/min. dim.}\}.$$

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In other words, for clans γ_1, γ_2 in $\mathcal{C}_{p,q}$, we have corresponding **B**-orbits $\mathcal{O}_{\gamma_1}$ and $\mathcal{O}_{\gamma_2}$. Then the Bruhat poset $(\mathcal{C}_{p,q}, \leq)$ is defined by the relation

$$\gamma_1 \leq \gamma_2 \iff \mathcal{O}_{\gamma_1} \subseteq \overline{\mathcal{O}_{\gamma_2}}.$$

Stemming from...

Theorem (Matsuki-Oshima '90)

Let $\mathcal{C}_{p,q}$ denote the set of (p,q)-clans. Then

$$\mathcal{C}_{p,q} \underset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \mathbf{B}\text{-orbits in } \mathsf{GL}_{p+q} / \mathsf{GL}_p \times \mathsf{GL}_q \\ \text{or} \\ \mathsf{GL}_p \times \mathsf{GL}_q\text{-orbits in } \mathsf{GL}_n / \mathbf{B} \end{array} \right\}.$$

Moreover, a clan is matchless if it consists only of $+/-$'s. Hence,

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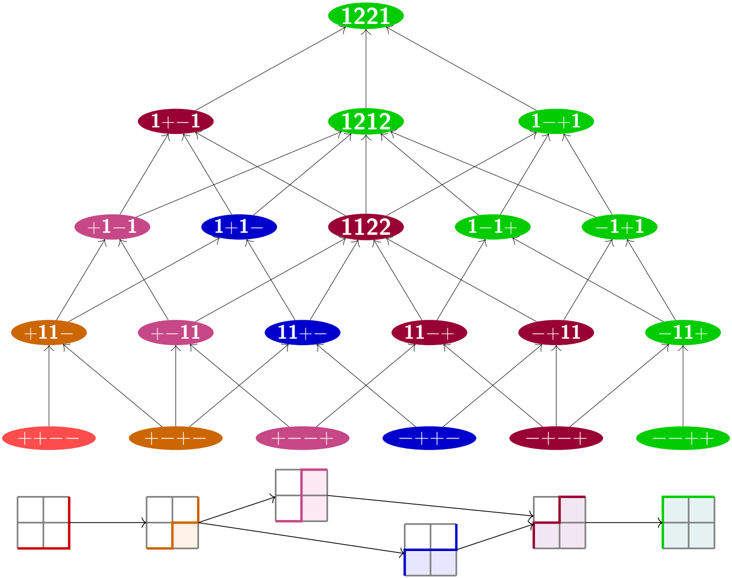
Shortcoming: There is no longer a unique minimal element...

Bruhat poset $\mathcal{C}_{2,2}$ (à la Wyser '16)



Sects

Def/Prop (Bingham-Can '20)

Let C_λ denote the Schubert cell of $\mathrm{GL}_{p+q}\,/\,\mathbf{P}$ associated to the partition $\lambda \in \binom{[p+q]}{p}$. Then the **sect** $\mathcal{C}^{\lambda}_{p,q}$ is the collection of clans γ whose corresponding orbits satisfy $\pi(\mathcal{O}_\gamma) = C_\lambda$ where $\pi : \mathbf{G}\,/\mathbf{L} \to \mathbf{G}\,/\,\mathbf{P}$ is the natural projection map.

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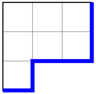
Combinatorially

(i) $\{\text{matchless/base clans } \tau\} \underset{1:1}{\longleftrightarrow} \{\text{sects/Schubert cells/partitions } \lambda\}$

$$\tau_\gamma = -+--++ \quad \longleftrightarrow$$



(ii) Label the symbols of γ as $\gamma = c_1 \cdots c_{p+q}$. For each pair $c_i = c_j \in \mathbb{N}$ with $i < j$, replace c_i by a $-$ symbol and c_j by a $+$ symbol.

$$\gamma = 1{+}{-}221 \quad \mapsto \quad -+--++ = \tau_\gamma$$

From sects to rooks

Let $R(\lambda)$ denote the set of **rook placements** of partition λ. For ρ, π in $R(\lambda)$, we say $\rho \leq \pi \iff rt_\rho \leq rt_\pi$ where $R(\lambda) \xrightarrow{rt} \mathbb{N}$ is a labeling of the boxes of $[\lambda]$ by the number of rooks weakly NW of each box.

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Bingham '21

There is an isomorphism between $R(\lambda)$ and $\mathcal{C}^\lambda_{p,q}$ as follows.

(i) Let the positions of the $-$ symbols in τ_γ be $i_1, \ldots, i_q$ and the positions of the $+$ symbols be $j_1, \ldots, j_p$ from left to right,

(ii) For each pair $c_{i_k} = c_{j_l} \in \mathbb{N}$ in γ, we place a rook in the square with northeast corner (k, l).

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$\gamma = 1{+}{-}221 \rightsquigarrow \tau_\gamma = -{+}{-}{-}{+}{+}$. The labels of τ_γ are $c_{i_1} c_{j_1} c_{i_2} c_{i_3} c_{j_2} c_{j_3}$. As $c_{i_1} = c_{j_3}$, we have a ♖ in the box with NE corner $(1, 3)$ and so on.

Partial permutations

The partial permutation associated to a clan $\gamma \in \mathcal{C}_{p,q}$ is the function $\phi_\gamma : [q] \to [p] \cup \{0\}$ defined by labeling the positions of the $-$ symbols in τ_γ by $i_1, \ldots, i_q$ and the positions of the $+$ symbols as $j_1, \ldots, j_p$ in *ascending* order.

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(2) Modify γ by iteratively replacing all 1212 patterns by 1221 patterns to obtain a clan which we call $\hat{\gamma}_0 \in \mathcal{C}^\lambda_{p,q}$ and which has symbols $\hat{c}_1 \cdots \hat{c}_{p+q}$.

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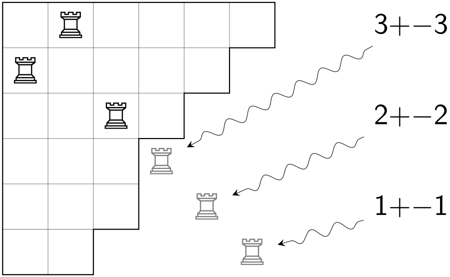
(4) Repeat the procedure of the previous step on $\hat{\gamma}_1$, and so on until we obtain a clan $\hat{\gamma}_s$ which is free of $1{+}{-}1$ patterns.

Example: *hidden* rooks

$\gamma = 12{+}3{+}{+}{-}3{-}1{-}2 \longmapsto$



The hidden rook associated to the pattern $2{+}{-}2 = \hat{c}_{i_2}\hat{c}_{j_2}\hat{c}_{i_5}\hat{c}_{j_5}$ is obtained from $\hat{\gamma}_1$, after changing the 1212 pattern to 1221 and deleting the symbols $3{+}{-}3 = \hat{c}_{i_3}\hat{c}_{j_3}\hat{c}_{i_4}\hat{c}_{j_4}$.

$$\gamma = 12{+}3{+}{+}{-}3{-}1{-}2 \mapsto \phi_\gamma = (5,6,4,3,2,1).$$

Coverings on $\mathcal{C}^{\lambda}_{p,q}$ (à la Wyser '16)

Lemma (Bingham-D, '25)

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Labelling on $\mathcal{C}_{p,q}^{\lambda}$

Definition (Bingham-D, '25)

Suppose that $\gamma \lessdot \tau$ is a covering relation where γ and the covering moves are as before. Then we label the covering relation (γ, τ) in $C(\mathcal{C}_{p,q}^{\lambda})$ with an element of $\mathbb{N}^2$ as follows.

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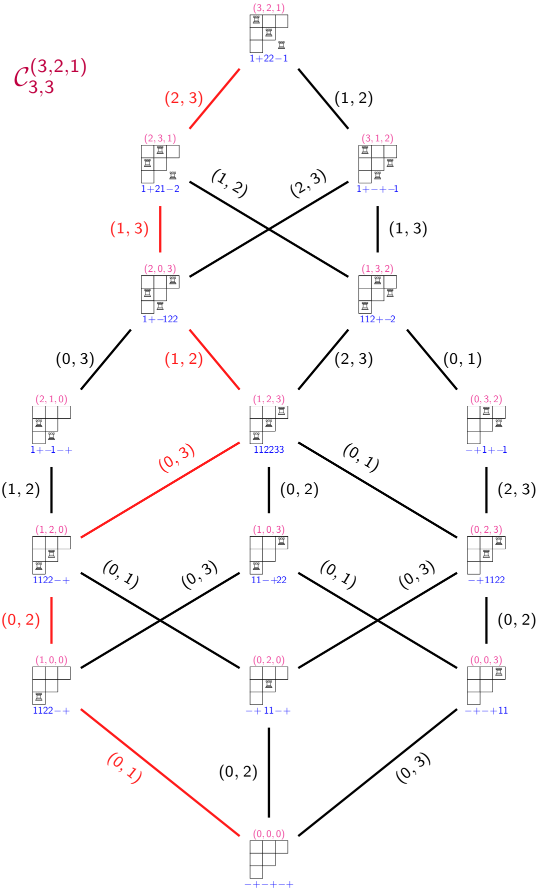
Theorem (Bingham-D' 25)

The Bruhat order on $GL_{p+q}/GL_p \times GL_q$ restricted to any sect $\mathcal{C}_{p,q}^{\lambda}$ is an *EL*-shellable poset.



$\mathcal{C}_{3,3}^{(3,2,1)}$
(3, 2, 1)
1+22−1
(2, 3)
(1, 2)
(2, 3, 1)
1+21−2
(3, 1, 2)
1+−+−1
(1, 2)
(2, 3)
(1, 3)
(1, 3)
(2, 0, 3)
1+−122
(1, 3, 2)
112+−2
(0, 3)
(1, 2)
(2, 3)
(0, 1)
(2, 1, 0)
1+−1−+
(1, 2, 3)
112233
(0, 3, 2)
−+1+−1
(0, 3)
(0, 1)
(1, 2)
(0, 2)
(2, 3)
(1, 2, 0)
1122−+
(1, 0, 3)
11−+22
(0, 2, 3)
−+1122
(0, 1)
(0, 3)
(0, 1)
(0, 3)
(0, 2)
(0, 2)
(1, 0, 0)
1122−+
(0, 2, 0)
−+11−+
(0, 0, 3)
−+−+11
(0, 1)
(0, 2)
(0, 3)
(0, 0, 0)
−+−+−+

Open questions...

(1) How could we conclude that $\mathcal{C}_{p,q}$ is shellable?

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Thank You/Gracias/Obrigado 😄

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"If you spend your time chasing butterflies, they will just fly away. But if you build a beautiful garden, the butterflies will come" Mario Quintana.