The Second Cohomology of Hess(X,H) and the Dot Action in types B/C

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Interactions between Geometry, Combinatorics, and Flag Varieties
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A *flag variety* is the quotient G/B for G an (appropriate) algebraic group and B a Borel subgroup.

$$\text{Type A} \longrightarrow \mathrm{Flag}(\mathbb{C}^n) = \{(V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)\}$$

A *Hessenberg variety* is a subvariety of G/B defined using $X \in \mathfrak{g}$ and a B-stable $H \subset \mathfrak{g}$:

$$\mathrm{Hess}(X, H) = \{gB \mid \mathrm{ad}(g^{-1}).X \in H\}$$

In type A, H naturally defines $h\colon [n] \to [n]$ such that

$$\text{Type A} \longrightarrow \mathrm{Hess}(X, h) = \{(V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \mid XV_i \subset V_{h(i)}\}$$

We focus on types B and C, and when X is regular semisimple.

Goal: Understand $H_T^*(\mathrm{Hess}(X,H))$ and $H^*(\mathrm{Hess}(X,H))$

Motivation: There's a natural (dot) action of the Weyl group W on $H_T^*(\mathrm{Hess}(X,H))$ and $H^*(\mathrm{Hess}(X,H))$ with *many* connections:

- Chromatic Quasisymmetric Functions [7, 3, 5]
- Unicellular LLT polynomials [2, 1]
- Hecke Algebra Characters [8]
- Monodromy [3]

We want to:

- generalize results from type A to other Lie types [6]
- See which special behaviors from type A might hold in all Lie types.

Our result is in types B and C, where the Weyl group is the *signed permutations* $\mathfrak{W}_n$.

Main Theorem

The character of the dot action representation of W *on* $H^2(\mathrm{Hess}(X, H))$ *is a nonnegative sum of characters of the following representations:*

- *The trivial representation*
- *The action on cosets of* $\mathfrak{S}_k \times \mathfrak{W}_{n-k}$
- *The action on cosets of* $\mathfrak{W}_1 \times \mathfrak{W}_{n-1}$
- *The 1-dim'l representation* δ

Fact

Each of these representations have characters that are $h_\lambda(x)h_\mu(y)$*-positive under the Frobenius character map to* $\Lambda(x, y)$.

GKM Theory

The T-equivariant cohomology of a GKM (Goresky-Kottwitz-MacPherson) variety can be given a combinatorial description [4].

GKM Varieties include:

- Schubert varieties,
- Regular semisimple Hessenberg varieties,
- and *many* more.

A key combinatorial construction is the GKM-graph, where

$$\{\text{Vertices}\} \longleftrightarrow \{\text{Torus Fixed Points}\}$$

$$\{\text{Edges}\} \longleftrightarrow \left\{\begin{matrix}\text{One Dimensional} \\ \text{Torus Orbits}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{Edge} \\ \text{Labels}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Torus} \\ \text{Weights}\end{matrix}\right\}$$

GKM Theory and Hess(X,H)

For (regular semisimple) $\mathrm{Hess}(X, H)$ in types B and C, this is:

$$\{\text{Vertices}\} \longleftrightarrow \{\text{Signed Permutations}\}$$

$$\{\text{Edges}\} \longleftrightarrow \left\{(w, ws_\alpha) \mid \alpha \in -(H \cap \Phi^-)\right\}$$

$$\left\{\begin{matrix}\text{Edge} \\ \text{Labels}\end{matrix}\right\} \longleftrightarrow \{(w, ws_\alpha) \mapsto w(\alpha)\}$$

We can turn this problem fully combinatorial by translating s_α to a *signed transposition*.

Signed Permutations

Let $\bar{i} \coloneqq -i$ and $[\overline{n}] \coloneqq \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$

The *signed permutations* $w \in \mathfrak{W}_n$ are bijections $w\colon [\overline{n}] \to [\overline{n}]$ such that $w(\bar{i}) = \overline{w(i)}$.

One-line notation is $\longrightarrow [w(1), w(2), \ldots, w(n)]$

$$\mathfrak{W}_2 \text{ contains: } \begin{matrix} [1,2], & [1,\bar{2}], & [\bar{1},2], & [\bar{1},\bar{2}] \\ [2,1], & [2,\bar{1}], & [\bar{2},1], & [\bar{2},1] \end{matrix}.$$

The *signed transpositions* are:

- $(i,j) \coloneqq (i,j)(\bar{i},\bar{j})$
- $(i,\bar{j}) \coloneqq (i,\bar{j})(\bar{i},j)$
- $(i,\bar{i})$

We translate from roots to transpositions by how the corresponding reflection actions on $e_1, \ldots, e_n$

Type B roots	Vector	Type C roots	Vector	Transposition
[100]	$(1,-1,0)$	[100]	$(1,-1,0)$	$(1,2)$
[010]	$(0,1,-1)$	[010]	$(0,1,-1)$	$(2,3)$
[001]	$(0,0,1)$	[001]	$(0,0,2)$	$(3,\overline{3})$
[110]	$(1,0,-1)$	[110]	$(1,0,-1)$	$(1,3)$
[012]	$(0,1,1)$	[011]	$(0,1,1)$	$(2,\overline{3})$
[011]	$(0,1,0)$	[021]	$(0,2,0)$	$(2,\overline{2})$
[112]	$(1,0,1)$	[111]	$(1,0,1)$	$(1,\overline{3})$
[122]	$(1,1,0)$	[121]	$(1,1,0)$	$(1,\overline{2})$
[111]	$(1,0,0)$	[221]	$(2,0,0)$	$(1,\overline{1})$

Roots and Transpositions for $\mathfrak{W}_3$ in both types.

The GKM Theorem for Hess(X,H)

Let $S(H)$ be the set of signed transpositions that correspond to H. Then $H_T^*(\mathrm{Hess}(X, H))$ is isomorphic to

$$\left\{ \boldsymbol{\rho} \in \prod_{w \in \mathfrak{W}_n} \mathbb{C}[x_\bullet] \;\middle|\; \boldsymbol{\rho}(w) - \boldsymbol{\rho}(w(i,j)) \in \left\langle x_{w(i)} - x_{w(j)} \right\rangle \text{ if } (i,j) \in S(H) \right\},$$

whose elements are called *splines*. This is

- A module over $\mathbb{C}[x_\bullet] := \mathbb{C}[x_1, ..., x_n]$ by pointwise-multiplication
- A ring by pointwise addition and multiplication
- A module over $\mathfrak{W}_n$ by $w \cdot \boldsymbol{\rho}(v) = w\boldsymbol{\rho}(w^{-1}v)$

$$S(H) = \{(1,2),(2,\bar{2})\}$$

$$\begin{tikzcd}
 & {[\bar{1},\bar{2}]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[\bar{1},2]} \arrow[d, dash] & & {[\bar{2},\bar{1}]} \arrow[d, dash] \\
{[2,\bar{1}]} \arrow[d, dash] & & {[\bar{2},1]} \arrow[d, dash] \\
{[2,1]} \arrow[dr, dash] & & {[1,\bar{2}]} \arrow[dl, dash] \\
 & {[1,2]} &
\end{tikzcd}$$

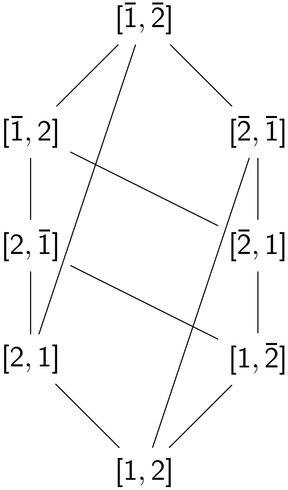
Examples in $\mathfrak{W}_2$ (Graph 1)

$$S(H) = \{(1,2),(2,\bar{2})\}$$

$$\begin{tikzcd}
 & {[\bar{1},\bar{2}]} \arrow[dl, dash, "x_2"'] \arrow[dr, dash, "x_1 - x_2"] & \\
{[\bar{1},2]} \arrow[d, dash, "x_1 + x_2"'] & & {[\bar{2},\bar{1}]} \arrow[d, dash, "x_1"] \\
{[2,\bar{1}]} \arrow[d, dash, "x_1"'] & & {[\bar{2},1]} \arrow[d, dash, "x_1 + x_2"] \\
{[2,1]} \arrow[dr, dash, "x_1 - x_2"'] & & {[1,\bar{2}]} \arrow[dl, dash, "x_2"] \\
 & {[1,2]} &
\end{tikzcd}$$

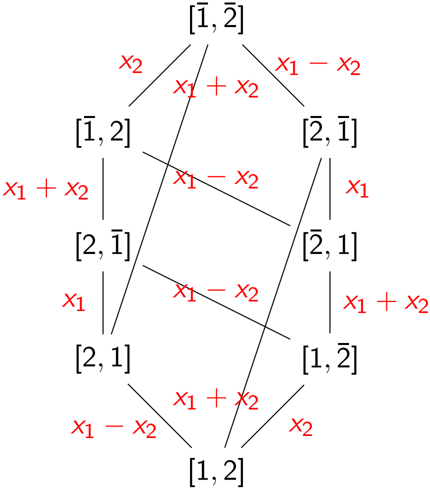
Examples in $\mathfrak{W}_2$ (Graph 2)

$$S(H) = \{(1,2), (2,\bar{2}), (1,\bar{2})\}$$



Examples in $\mathfrak{W}_2$ (Graph 2)

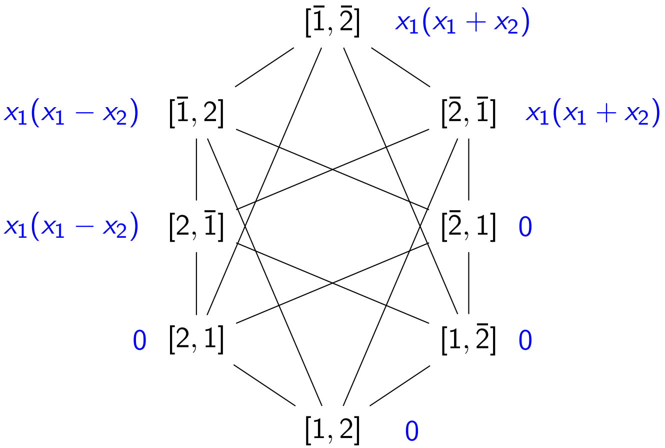
$$S(H) = \left\{(1,2),(2,\bar{2}),(1,\bar{2})\right\}$$



Examples in $\mathfrak{W}_2$ (Splines)

$$
\begin{tikzcd}[column sep=small]
 & {[\bar{1},\bar{2}]}\ \color{blue}{x_2} \arrow[dl, no head] \arrow[dr, no head] & \\
\color{blue}{0}\ {[\bar{1},2]} \arrow[d, no head] & & {[\bar{2},\bar{1}]}\ \color{blue}{x_1} \arrow[d, no head] \\
\color{blue}{-x_1-x_2}\ {[2,\bar{1}]} \arrow[d, no head] & & {[\bar{2},1]}\ \color{blue}{0} \arrow[d, no head] \\
\color{blue}{x_1-x_2}\ {[2,1]} \arrow[dr, no head] & & {[1,\bar{2}]}\ \color{blue}{0} \arrow[dl, no head] \\
 & {[1,2]}\ \color{blue}{0} &
\end{tikzcd}
$$

Examples in $\mathfrak{W}_2$ (Splines)



The dot action moves polynomials around and acts on the indices.

$$(1,\bar{2})\cdot\ \begin{tikzcd}[column sep=small]
 & {[\bar{1},\bar{2}]\ x_2} \arrow[dl, dash] \arrow[dr, dash] & \\
{0\ [\bar{1},2]} \arrow[d, dash] & & {[\bar{2},\bar{1}]\ x_1} \arrow[d, dash] \\
{-x_1-x_2\ [2,\bar{1}]} \arrow[d, dash] & & {[\bar{2},1]\ 0} \arrow[d, dash] \\
{x_1-x_2\ [2,1]} \arrow[dr, dash] & & {[1,\bar{2}]\ 0} \arrow[dl, dash] \\
 & {[1,2]\ 0} &
\end{tikzcd}
\ = \
\begin{tikzcd}[column sep=small]
 & {[\bar{1},\bar{2}]\ -x_2+x_1} \arrow[dl, dash] \arrow[dr, dash] & \\
{x_1+x_2\ [\bar{1},2]} \arrow[d, dash] & & {[\bar{2},\bar{1}]\ 0} \arrow[d, dash] \\
{0\ [2,\bar{1}]} \arrow[d, dash] & & {[\bar{2},1]\ 0} \arrow[d, dash] \\
{-x_1[2,1]} \arrow[dr, dash] & & {[1,\bar{2}]\ 0} \arrow[dl, dash] \\
 & {[1,2]-x_2} &
\end{tikzcd}$$

Let $t_i := (i, i+2)$ for $i \in [n-2]$, $t_{n-1} = (n-1, \overline{n-1})$, $t_n = (n-1, \overline{n})$, and $\mathcal{T} := \{t_1, ..., t_n\}$.

Lemma

The 2^{nd} equivariant $H^2_T(\mathrm{Hess}(X, H))$ and ordinary $H^2(\mathrm{Hess}(X, H))$ cohomology, as well as the dot action on both, is entirely determined by $S(H) \cap \mathcal{T}$.

•allows us to unify the type-B and type-C calculations

•gives a condition for $H^2_T(\mathrm{Hess}(X, H)) = H^2_T(\mathrm{Hess}(X, H'))$.

Main Theorem

The character of the dot action representation on $H^2(\mathrm{Hess}(X, H))$ is determined from the following data of $S(H) \cap \mathcal{T}$:

$$\begin{array}{rcl} \textit{Trivial Representation} & \longleftrightarrow & |S(H) \cap \mathcal{T}| \quad (+1) \\ \textit{Action on } \mathfrak{S}_k \times \mathfrak{W}_{n-k} & \longleftrightarrow & \{t_{k-1}, t_k\} \cap S(H) = \emptyset \\ \textit{Action on } \mathfrak{S}_1 \times \mathfrak{W}_{n-1} & \longleftrightarrow & \{t_{k-1}, t_k\} \cap S(H) = t_{k-1} \\ \textit{Action on } \mathfrak{W}_1 \times \mathfrak{W}_{n-1} & \longleftrightarrow & t_n \notin S(H) \\ \delta & \longleftrightarrow & \{t_{n-1}, t_n\} \cap S(H) = t_n \end{array}$$

Looking Forward

Many questions remain.

- Type D
- Total description of trivial (flag-isomorphic) cohomology?
- First non-trivial (non-flag isomorphic) cohomology?
- Bases for $H^*_T(\mathrm{Hess}(X, H))$?
- Graph coloring schema?
- Is the Frobenius Character always $h_\lambda(x)h_\mu(y)$-positive?

Thank you!

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