Orbit structures and complexity in Schubert varieties and Richardson varieties

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Saint Louis University, 2025 Fall Central Sectional
Oct 2025

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The B-orbits in GL_n/B are parametrized by $\mathfrak{S}_n$.

For $w \in \mathfrak{S}_n$, $X_w^\circ := BwB$ is the Schubert cell. Its Zariski closure is the Schubert variety,

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These are well studied varieties with a rich combinatorial structure.

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Richardson varieties

Let $u, v \in \mathfrak{S}_n$. The Richardson variety $\mathcal{R}_{u,v}$ is

$$\mathcal{R}_{u,v} := X_v \cap X^u$$

.

The Bruhat order

The Bruhat order on $\mathfrak{S}_n$ is the partial order $\leq$ induced by B-orbit closure containment in GL_n/B; that is, for $u, v \in \mathfrak{S}_n$

$$u \leq v \iff X_u \subseteq X_v$$

Let $t_{i,j} := (ij) \in \mathfrak{S}_n$ and $s_i := t_{i,i+1}$. Combinatorially, the Bruhat order is the transitive closure of

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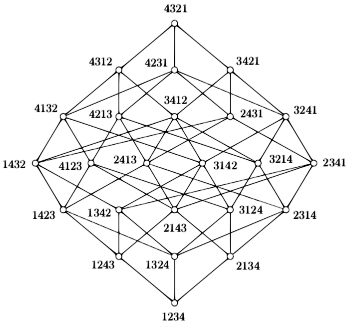

The Bruhat order on $\mathfrak{S}_4$.

Much of the geometric structure of GL_n/B is encoded in the combinatorics of the Bruhat order.

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Levi subgroups (and their Borel subgroups)

A standard parabolic subgroup of GL_n is a subgroup containing B.
For each $I \subseteq [n-1]$, there is a standard parabolic subgroup P_I with

$$P_I = L_I \ltimes U_I,$$

where U_I is its unipotent radical, and L_I is a reductive group called a Levi subgroup.

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When $I = [n-1]$, $P_I = L_I = GL_n$. In GL_n/B the Levi-Borel orbits are the Schubert cells.

When $I = \emptyset$, $L_I = T$. And we are studying T-orbits and T-orbit closures in X_w or GL_n/B.

Orbits of other groups in the flag variety and its subvarieties

Orbit complexity

Let G be a reductive algebraic group and B_G a Borel subgroup of G.

The G-complexity of a normal G-variety X, $c_G(X)$, is the minimum codimension of a B_G-orbit in X.

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Fact 1: The are a finite number of B_G-orbits if and only if $c_G(X) = 0$; in this case X is G-spherical.

Fact 2: A T-spherical variety is a toric variety.

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Complexity in the literature:

(I) Torus orbits ($T = L_\emptyset$):
- When is a Schubert variety is a a toric variety? (Karuppuchamy '13)
- When is a Richardson variety a toric variety? (Lee-Matsuda-Park '21, Can-Saha '23)
- What is the complexity of the torus action on a Schubert variety (or Richardson variety)? Type A: (Lee-Matsuda-Park '21, Donten Bury-Escobar-Portakal '23)

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(II) Levi-Borel orbits:
- When is the flag variety L_I-spherical? (Magyar-Weyman-Zelevinsky '99, Stembridge '03)
- When is a Schubert variety L_I-spherical? (Hodges-Yong '21, Gao-Hodges-Yong '22 & '23, Can-Saha '23)

Algebraic dimension of a Bruhat interval

The (undirected) Bruhat graph on $\mathfrak{S}_n$ is the graph Γ with vertex set $\mathfrak{S}_n$ and edges $w \sim t_{i,j}w$ for all $w \in \mathfrak{S}_n$ and $1 \leq i < j \leq n$.

Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$.

For each edge $w \sim t_{i,j}w$, we say that it has weight $e_i - e_j$, and write $\mathrm{wt}(w, t_{i,j}w) = e_i - e_j$.

For $u \leq v$, let $\Gamma(u, v)$ be the Bruhat graph restricted to the vertex set $[u, v]$.

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Algebraic dimension

For $u \leq v$, let $\mathrm{AD}(u, v)$ be the $\mathbb{R}$-span of all edge weights in $\Gamma(u, v)$, i.e.

$$\mathrm{AD}(u, v) = \mathrm{span}_{\mathbb{R}}\{\mathrm{wt}(x, y) \mid u \leq x < y \leq v\}.$$

Let $\mathrm{ad}(u, v) = \dim \mathrm{AD}(u, v)$ be the algebraic dimension of the Bruhat interval $[u, v]$.

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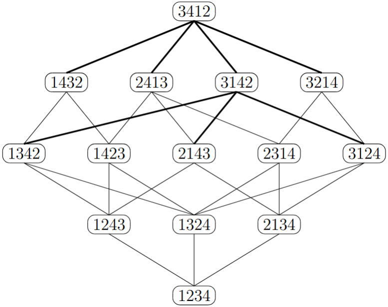
Proposition (Gao-H). $\mathrm{AD}(u, v)$ has the following properties.

1. For any saturated chain $u = w^{(0)} \lessdot w^{(1)} \lessdot \cdots \lessdot w^{(\ell-1)} \lessdot w^{(\ell)} = v$,
$$\mathrm{AD}(u, v) = \mathrm{span}_{\mathbb{R}}\{\mathrm{wt}(w^{(i)}, w^{(i+1)}) \mid i = 0, \ldots, \ell - 1\}.$$
2. For any $w \in [u, v]$, $\mathrm{AD}(u, v)$ is spanned by the weights of all cover relations incident to w inside $[u, v]$.

Algebraic dimension example

Let $u = 1234$ and $v = 3412$. The interval $[1234, 3412]$ in the Bruhat order is given below with cover relations bolded.



The weights from 3412 are $e_1 - e_3$, $e_2 - e_3$, $e_1 - e_4$ and $e_2 - e_4$ (from left to right) while the weights from 3142 are $e_1 - e_3$, $e_2 - e_3$ and $e_2 - e_4$ (from left to right on the bottom) and $e_1 - e_4$ (on the top). The same linear space is spanned by these two sets of weights.

Thus $\mathrm{ad}(1234, 3412) = 3$.

Torus complexity in Richardson varieties

Theorem (Gao-H). The T-complexity of the Richardson variety equals

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The cardinality of $\mathrm{Supp}(w)$ is written as $\mathrm{supp}(w) = |\,\mathrm{Supp}(w)|$.

Corollary (Gao-H). The T-complexity of the Schubert variety X_w equals

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Even better, this leads to a formula for the L_I-complexity of X_w.

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Theorem (Gao-H). Suppose L_I acts on the Schubert variety X_w. Then

$$c_{L_I}(X_w) = \ell(d) - \mathrm{supp}(d).$$

Future work

Generalized Bruhat orders: Given a Levi-Borel B_I such that B_I has a finite number of orbits in X_w, can we give an combinatorial indexing set for these orbits. And can we describe the partial order on this set induced by orbit closure containment?

Number of B_I orbits in general? We can say when there will be a finite number of B_I-orbits in X_w when L_I acts on X_w. But what about when L_I does not act? This is an open problem.

Thank you!