Staircase diagrams, pattern avoidance, and smooth Schubert varieties

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Let $n \in \mathbb{Z}_+$ and $\mathbb{C}^n = \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$.

Flag variety:

$$\mathrm{Fl}(n) := \{V_\bullet = (V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n) \mid \dim(V_i) = i\}$$

For any permutation $w \in S_n$, define the coordinate flag $E_\bullet^w \in \mathrm{Fl}(n)$ by

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Bruhat decomposition:
Let B denote $n \times n$ invertible upper triangular matrices. Then

$$\mathrm{Fl}(n) = \bigsqcup_{w \in S_n} B \cdot E_\bullet^w$$

Schubert variety:

$$X(w) := \overline{B \cdot E_\bullet^w}.$$

Remark: The geometry of Schubert varieties plays an important role in combinatorics and representation theory.

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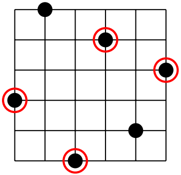
Pattern avoidance: Let $m \leq n$. We say a permutation $w = w(1) \cdots w(n)$ *contains* the pattern $u = u(1) \cdots u(m)$ if there is a subsequence of w with the same relative order as u. Otherwise, w *avoids* the pattern u.

Example: $w = 416253 =$ 

contains the pattern 3412, but avoids the pattern 1234.

Conventions:

- The matrix entries of w mark the points $(i, w(i))$ (col $\rightarrow$, row $\downarrow$).
- $(1,1)$ represents the NW corner of the matrix.

Theorem (Lakshmibai-Sandhya 1990, Carrell 1994, Ryan 1987):
Let $w \in S_n$. The following are equivalent:

1. $X(w)$ is smooth.
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3. The Bruhat interval $[e, w]$ is rank symmetric.
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Enumeration (Haiman preprint, Bousquet-Mélou-Butler 2007):
Let

$$a_n := \#\{w \in S_n \mid X(w) \text{ is smooth}\}.$$

Then

$$\sum_{n \geq 0} a_n \, x^n = \frac{1 - 5x + 3x^2 + x^2\sqrt{1-4x}}{1 - 6x + 8x^2 - 4x^3}.$$

Alternate encoding of smooth permutations:

Let $[n-1] := \{1, 2, \ldots, n-1\}$ and

$$\mathcal{D} = \{B_1, \ldots, B_k\}$$

be a set of intervals in $[n-1]$ where if $B_i = [l_i, r_i]$, then

$$l_i < l_{i+1} \text{ and } r_i < r_{i+1} \text{ for all } i.$$

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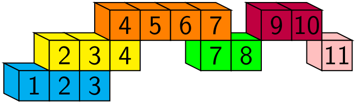
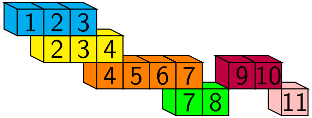
Definition: We say a partial order $\preceq$ on $\mathcal{D}$ is a *staircase diagram* on $[n-1]$ if the following conditions hold:

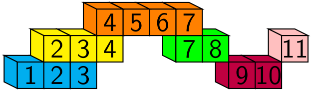
1. B_i covers B_j if and only if $j = i \pm 1$ and $B_i \cup B_j$ is connected.
2. If $B_i \preceq B_{i-1}, B_{i+1}$ or $B_i \succeq B_{i-1}, B_{i+1}$, then $B_{i-1} \cup B_{i+1}$ is disconnected.

Example: Let $n = 12$ and $\mathcal{D} = \{B_1, \ldots, B_6\}$:



Other staircase diagrams on $\mathcal{D} = \{B_1, \ldots, B_6\}$:





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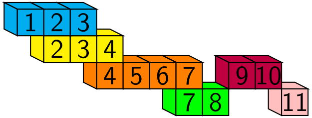

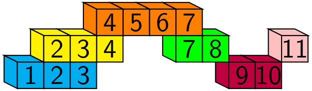

Permutation of a staircase diagram: Note that

$$S_n = \langle s_1, \ldots, s_{n-1} \rangle \text{ where } s_i := (i, i+1).$$

For any $J \subseteq [n-1]$, let u_J denote the *longest permutation* generated by $\{s_i \mid i \in J\}$. For $B \in \mathcal{D}$ define

$$J(B) := B \cap \left(\bigcup_{B' \prec B} B' \right).$$

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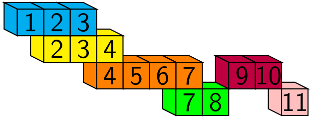

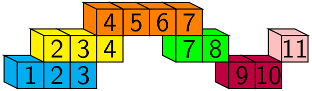

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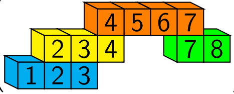
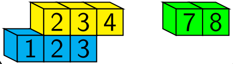
Define $\Lambda(\varnothing) :=$ id. Otherwise, let $\mathcal{D}$ be a staircase diagram and choose a maximal element $B \in \mathcal{D}$. Define

$$\Lambda(\mathcal{D}) := u_B \cdot u_{J(B)} \cdot \Lambda(\mathcal{D} \setminus \{B\}).$$

Example:

$\Lambda\Bigg($ 

$\Bigg) = u_{[4,7]} \cdot s_4 s_7 \cdot \Lambda\Bigg($ 

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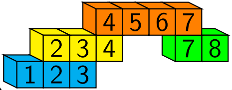
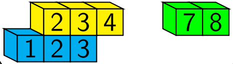
$\Lambda \Bigg($ 

$\Bigg) = u_{[4,7]} \cdot s_4 s_7 \cdot \Lambda \Bigg($ 

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Facts:

- $\Lambda(\mathcal{D})$ is well-defined.
- $u_B \cdot u_{J(B)}$ is maximal in $W_B \cap W^{J(B)}$.
- $\Lambda(\mathcal{D})^{-1} = \Lambda(\mathcal{D}^*)$ where $\mathcal{D}^*$ denotes the dual poset.
- The projection map

$$X(\Lambda(\mathcal{D})) \twoheadrightarrow X^{J(B)}(u_B)$$

is a $X(\Lambda(\mathcal{D} \setminus \{B\}))$-fiber bundle.

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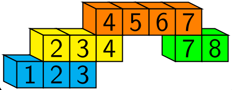
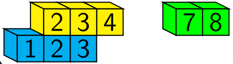
$\Lambda\Bigg($ 

$\Bigg) = u_{[4,7]} \cdot s_4 s_7 \cdot \Lambda\Bigg($ 

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Theorem (R-Slofstra 2017): The map Λ is a bijection:

$$\{\text{Staircase diagrams on } [n-1]\} \xrightarrow{\Lambda} \{w \in S_n \mid X(w) \text{ is smooth}\}$$

Remarks (R-Slofstra 2016, 2017, 2018):

- Versions of staircase diagrams exist for flag varieties in all finite types and affine type A.
- Haiman's enumeration (g.f.) of (rationally) smooth Schubert varieties generalizes to other types.

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1. $X(w)$ is a toric manifold.
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Question: What are staircase diagrams for Boolean permutations?

Staircase diagram of a Boolean permutation:

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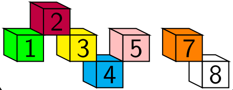

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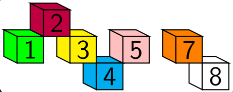
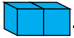
$\Lambda\Bigg($  $\Bigg) = (s_2)(s_1 s_3 s_5 s_7)(s_4 s_8) = 314625897.$

Compare statements:

- A smooth permutation is Boolean if and only if it avoids 321.
- A staircase diagram is Boolean if and only if it avoids .

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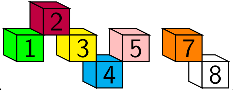
Λ () $= (s_2)(s_1 s_3 s_5 s_7)(s_4 s_8) =$ 314625897.

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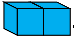
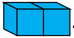
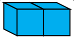
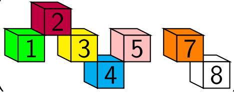
- A staircase diagram is Boolean if and only if it avoids .

TFAE:

1. The staircase diagram $\mathcal{D}$ avoids .
2. The permutation $\Lambda(\mathcal{D})$ avoids 321.

Moveover Λ () $=$ “321”

Staircase diagram of a Boolean permutation:

$$\Lambda\left(\right.$$ 

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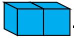
- A staircase diagram is Boolean if and only if it avoids .

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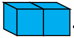
1. The staircase diagram $\mathcal{D}$ avoids .
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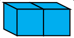
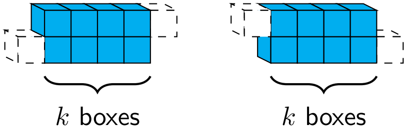
Moveover $\Lambda\left(\right.$  $\left.\right) =$ "321"

Question: Does this idea extend to other staircase shapes and patterns?

Staircase overlaps:



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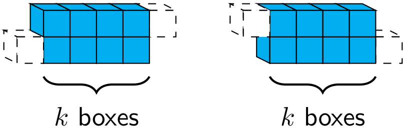
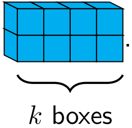
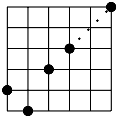
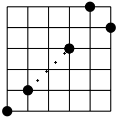


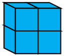
Theorem (Azam-R. 2025): The following are equivalent:

1. The staircase diagram $\mathcal{D}$ avoids .
2. The permutation $\Lambda(\mathcal{D})$ avoids

$$(k+2)(k+3)(k+1)k\cdots 21 \quad \text{and} \quad (k+3)(k+2)\cdots 4312.$$





Theorem (Azam-R. 2025): Let $k = 2$ (avoids ) and

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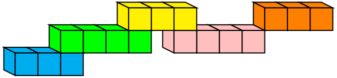
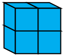
Proof:

Staircase overlaps are at most one: 

Theorem (Azam-R. 2025): Let $k = 2$ $\Big($avoids  $\Big)$ and

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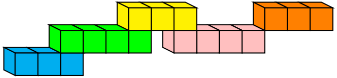
Staircase overlaps are at most one: 

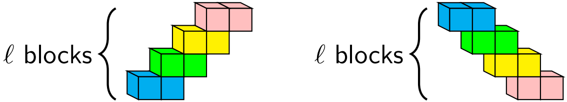
Curious connection:
Let

$$c_n = \#\{w \in S_n \mid w \text{ avoids } 4123 \text{ and } 4321\}.$$

Corollary: $a_n = c_n$ for all $n \geq 0$.

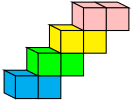
Proof: Same generating function (OEIS).

Strongly connected chains:



Strongly connected chains:

ℓ blocks



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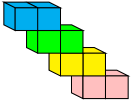
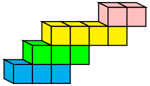
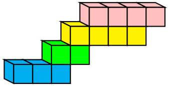
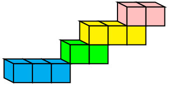


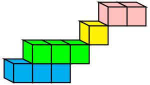
More strongly connected chains:





Weakly connected chains:

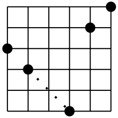


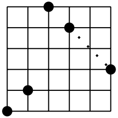


Theorem (Azam-R. 2025): The following are equivalent:

1. The staircase diagram $\mathcal{D}$ avoids strongly connected chains of length ℓ.
2. The permutation $\Lambda(\mathcal{D})$ avoids

$$34\cdots(\ell{+}1)(\ell{+}2)21 \quad \text{and} \quad (\ell{+}2)(\ell{+}1)12\cdots\ell.$$

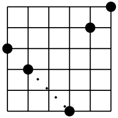


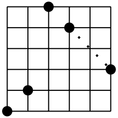


Theorem (Azam-R. 2025): The following are equivalent:

1. The staircase diagram $\mathcal{D}$ avoids strongly connected chains of length ℓ.
2. The permutation $\Lambda(\mathcal{D})$ avoids

$$34\cdots(\ell{+}1)(\ell{+}2)21 \quad \text{and} \quad (\ell{+}2)(\ell{+}1)12\cdots\ell.$$





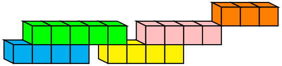
Theorem (Azam-R. 2025): Let $\ell = 3$ and

$$b_n = \#\{w \in S_n \mid w \text{ avoids } 3412,\ 4231,\ 34521,\ 54123\}.$$

Then

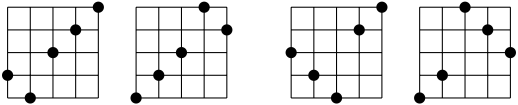
$$\sum_{n\geq 0} b_n\, x^n = \frac{1-5x+6x^2-4x^3}{1-6x+10x^2-8x^3+2x^4}.$$

Proof: Strongly connected chain length at most two:



Theorem (Gaetz-Gao 2020): Let $w \in S_n$. The following are equivalent (*polished permutations*):

1. The Bruhat interval $[e, w]$ is self-dual.
2. w is smooth and avoids 45321, 54312, 34521, 54123.



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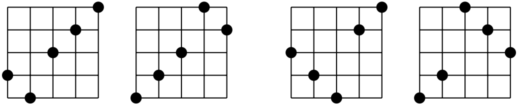
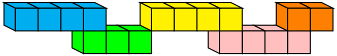
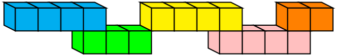


Observation: Polished permutations correspond to staircase diagrams such that:

1. Overlaps are at most one ($k = 2$).
2. Strongly connected chain length at most two ($\ell = 3$).





Theorem (Azam-R. 2025): Let

$$a_n = \#\{w \in S_n \mid w \text{ is polished}\}.$$

Then

$$\sum_{n \geq 0} a_n \, x^n = \frac{1 - 2x + x^2 + x^4}{1 - 2x + x^3}.$$

Thank you!