Affine Springer fibers and the Delta and Shuffle Theorems

AMS Central Sectional Meeting
St. Louis

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Joint work with Maria Gillespie and Eugene Gorsky

October 18, 2025

Springer fibers

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- $H^{\text{top}}(\mathrm{Fl}_x; \mathbb{Q}) \cong_{S_n} V^\lambda$ irreducible
- The graded S_n-module structure can be characterized using symmetric functions...

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(virtual) $\mathbb{Q}S_n$-modules $\xrightarrow{\text{Frob}}$ Symmetric functions

$$\mathbb{Q} \quad \leftrightarrow \quad h_n = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$\mathrm{sgn} \quad \leftrightarrow \quad e_n = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

V^λ (irreducible) $\quad \leftrightarrow \quad s_\lambda$

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$$V^\lambda \text{ (irreducible)} \quad \leftrightarrow \quad s_\lambda$$

If $V = \bigoplus_i V_i$ is a graded S_n-module, it has a *graded Frobenius character*,

$$\mathrm{Frob}(V; q) := \sum_i \mathrm{Frob}(V_i) q^i.$$

Frobenius characters

Thm (Hotta–Springer)

$\mathrm{Frob}(H^*(\mathrm{Fl}_x; \mathbb{Q}); q) = \widetilde{H}_\lambda(x; q)$ Hall–Littlewood polynomial

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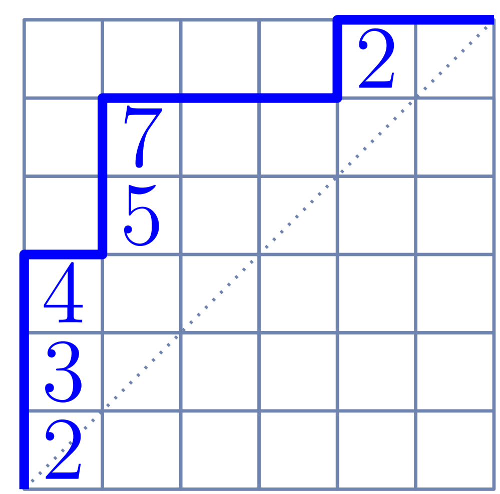
- It diagonalizes the **Macdonald polynomial** basis of $\Lambda_{q,t}$, but is difficult to compute on other bases.
- Astonishingly, the evaluation ∇e_n has a wonderful formula in terms of **word parking functions**:



Shuffle Theorem

Conjectured by Haglund–Haiman–Loehr–Remmel–Ulyanov.

Shuffle Theorem (Carlsson–Mellit, 2018)

$$\nabla e_n = \sum_{P \in \mathcal{WPF}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x^P.$$

Ex: $n = 6$

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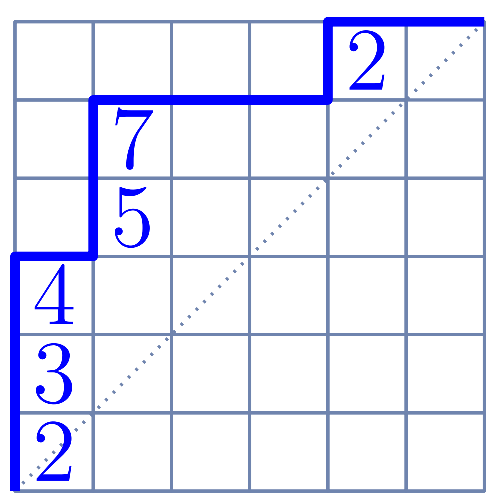

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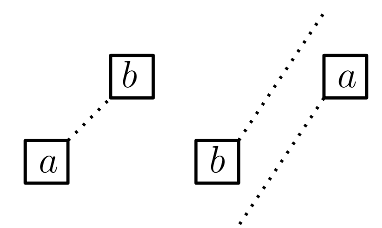

$(a < b)$

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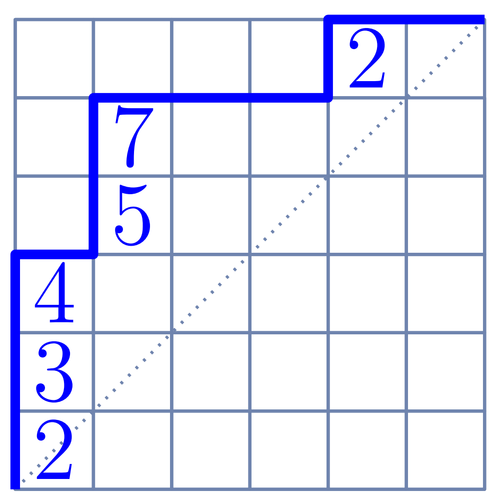
$P =$ 

$\mathrm{dinv}(P) = 3$

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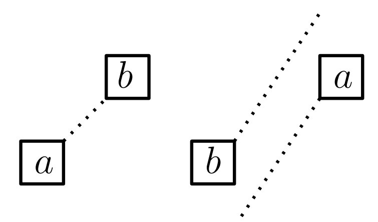

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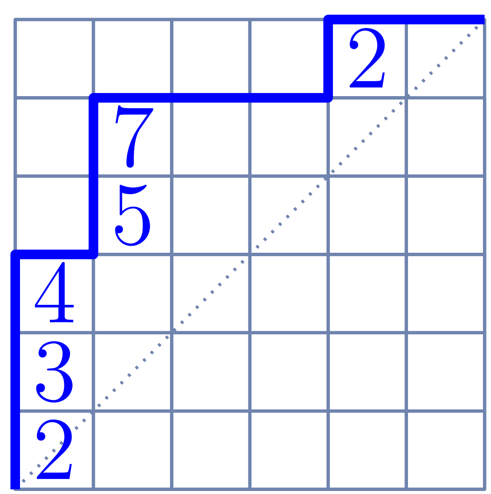

$\mathrm{dinv}(P) = 3$

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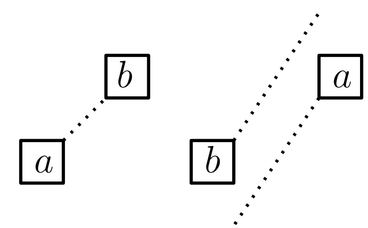

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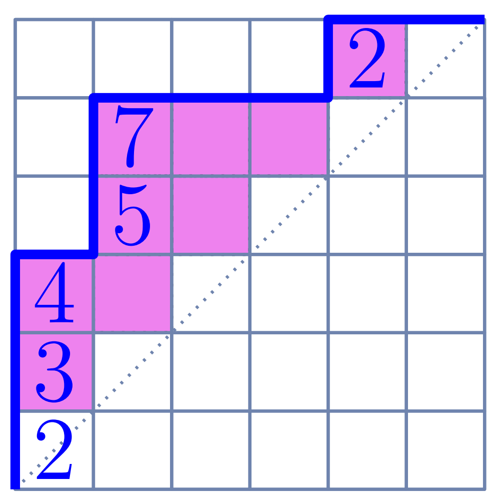

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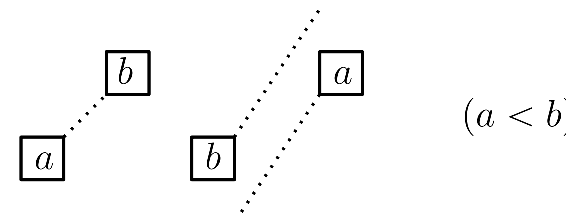

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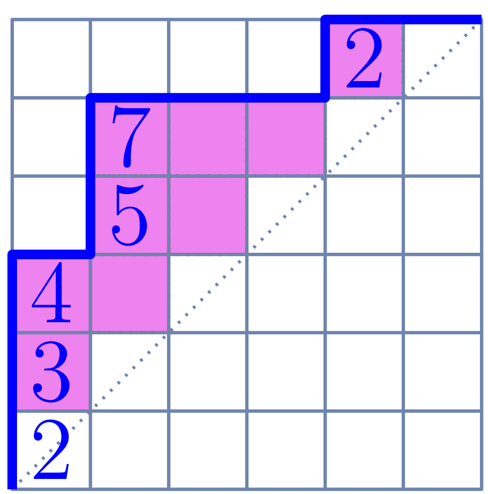

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$x^P = x_2^2 x_3 x_4 x_5 x_7$

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$\mathrm{Frob}(H_*(??); q, t) = \nabla e_n$

Yes, this symmetric function does come from a geometric S_n action!

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We must "upgrade" to affine Springer fibers to see the t grading.

Flags of lattices

$\mathcal{O} = \mathbb{C}[\![\epsilon]\!]$, $\mathcal{K} = \mathbb{C}(\!(\epsilon)\!)$

A **lattice** Λ is a $\mathcal{O}$-submodule of $\mathcal{K}^n$ of rank n.

Example: $n = 5$

$$\begin{array}{ccccc}
\vdots & & & & \vdots \\
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Hikita's Theorem

By Lusztig, there is a Springer action of S_n on $H_*(\widetilde{Fl}_\gamma; \mathbb{Q})$.

Let $\gamma = \left(\begin{array}{c|c} 0 & \epsilon^2 \\ \hline \epsilon I_{n-1} & 0 \end{array}\right)$.

Theorem (Hikita, 2012)

$$\mathrm{Frob}(H_*(\widetilde{Fl}_\gamma; \mathbb{Q}); q, t) = \nabla e_n.$$

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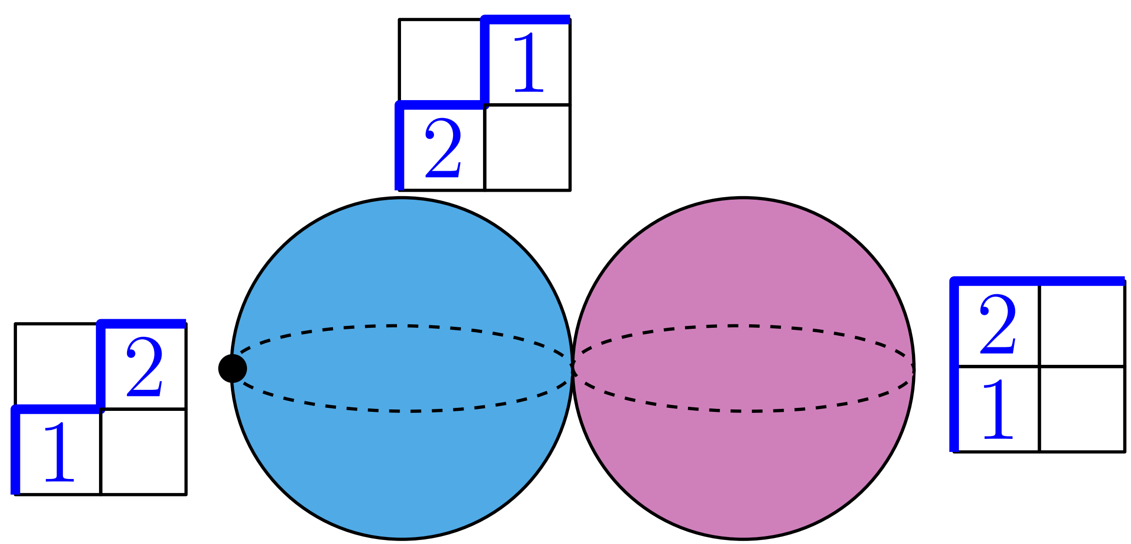
Ex: $n = 2$

$\widetilde{Fl}_\gamma \cong$



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- This result is for the SL_n-version of $\widetilde{Fl}_\gamma$.
- q grading = (halved) homological co-degree.
- t grading comes from a filtration of $\widetilde{Gr}_\gamma$.

Rectangular Shuffle Theorem

Let n, m, k be positive integers such that $gcd(n, m) = 1$.

$\mathcal{WPF}_{kn,km} = \{(kn) \times (km)$ word parking functions$\}$

Rectangular Shuffle Thm (Mellit, 2021)

$$E_{kn,km} \cdot 1 = (-1)^{k(m+1)} \sum_{P \in \mathcal{WPF}_{kn,km}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x^P.$$

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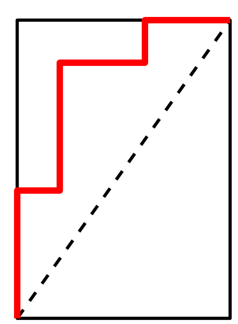
Left side (algebraic): An Elliptic Hall Algebra element $E_{kn,km}$ acting on 1

Right side (combinatorial):



dinv is now computing using inversions along lines of slope n/m

Affine Springer fiber for Rectangular Shuffle

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Theorem (Gillespie–Gorsky–G.)

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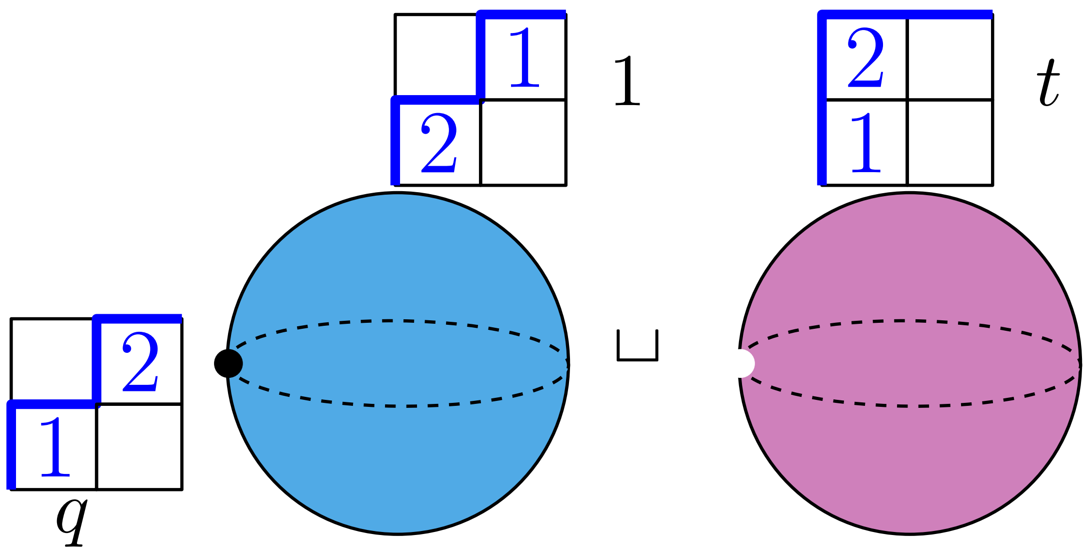
- t grading is by connected components!

Ex: $n = k = 2$



Geometry to combinatorics

Theorem (Gillespie–Gorsky–G.)

$$\mathrm{Frob}(H_*(\widetilde{\mathrm{Fl}}_\gamma^{+,0};\mathbb{Q});q,t) = E_{kn,k}\cdot 1$$

The action of γ on basis vectors:

$$\begin{tikzcd}
e_1 \arrow[r] & e_4 \arrow[r] & e_7 \arrow[dr] & e_2 \arrow[r] & e_5 \arrow[r] & e_8 \arrow[dr] & e_3 \arrow[r] & e_6 \arrow[r] & e_9 \\
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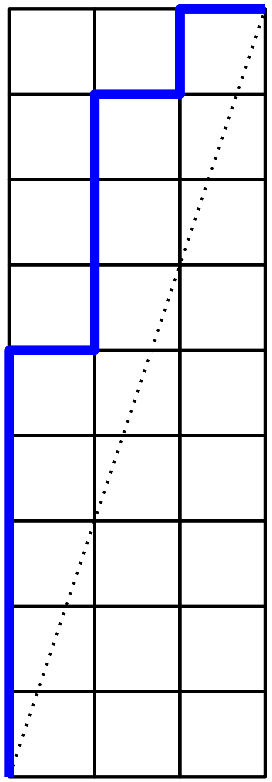

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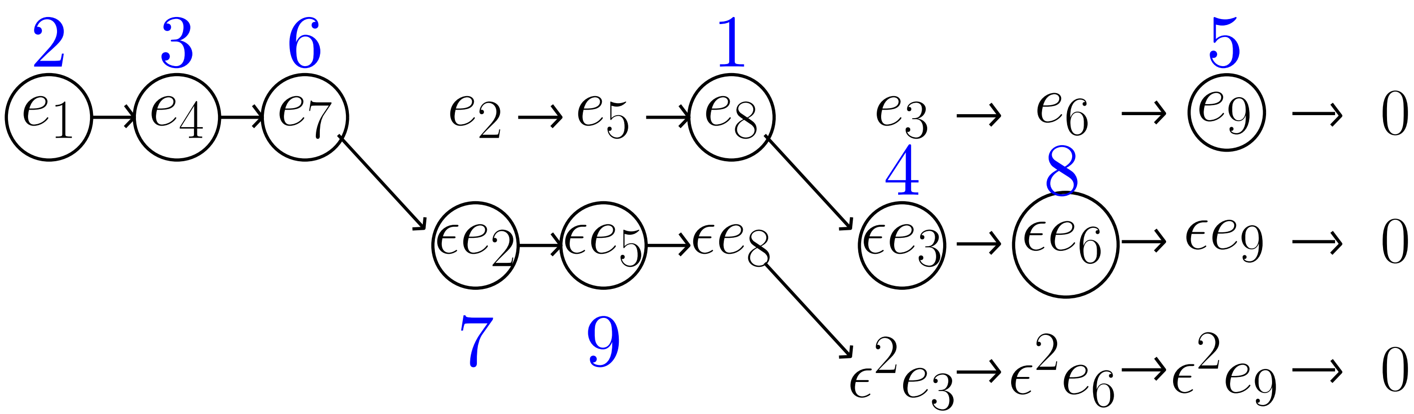

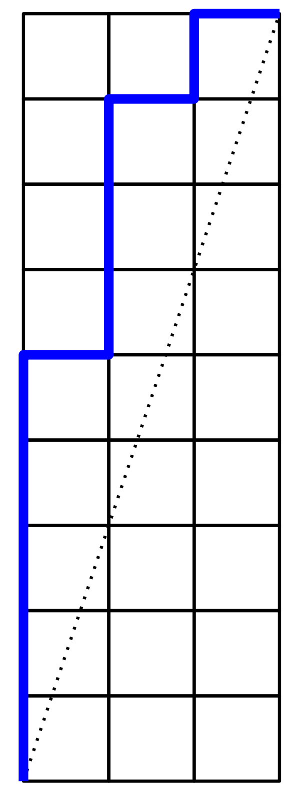

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Flag of lattices $\leftrightarrow$ Labeling of the circled elements

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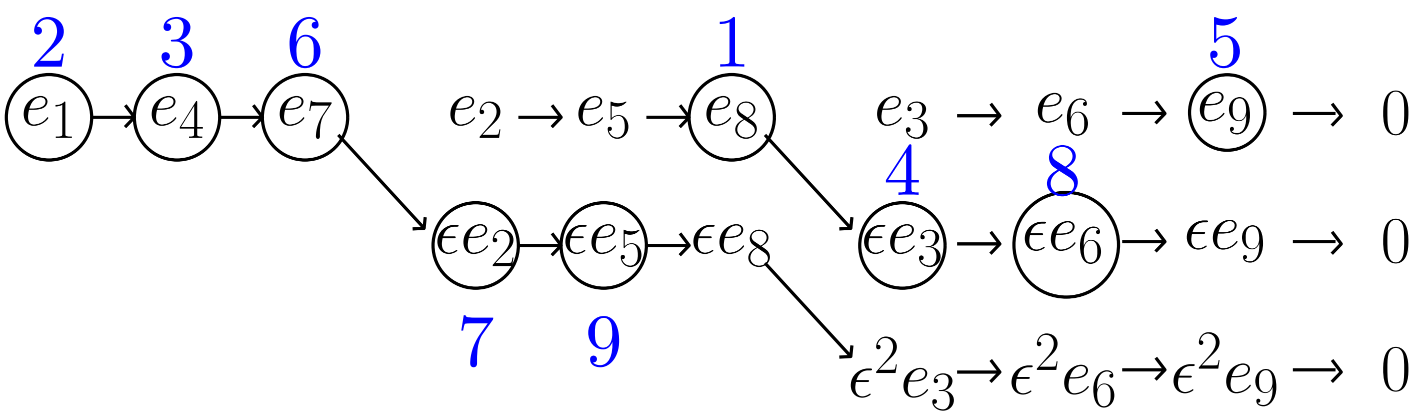

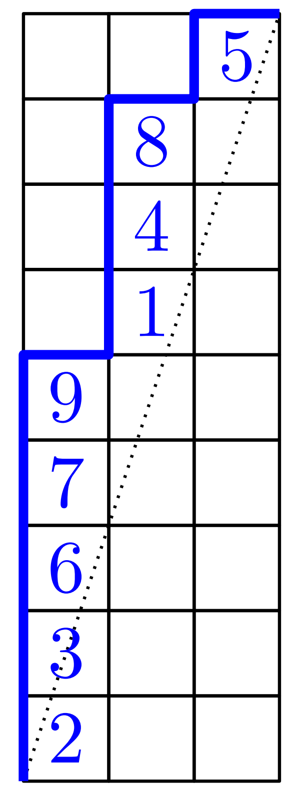

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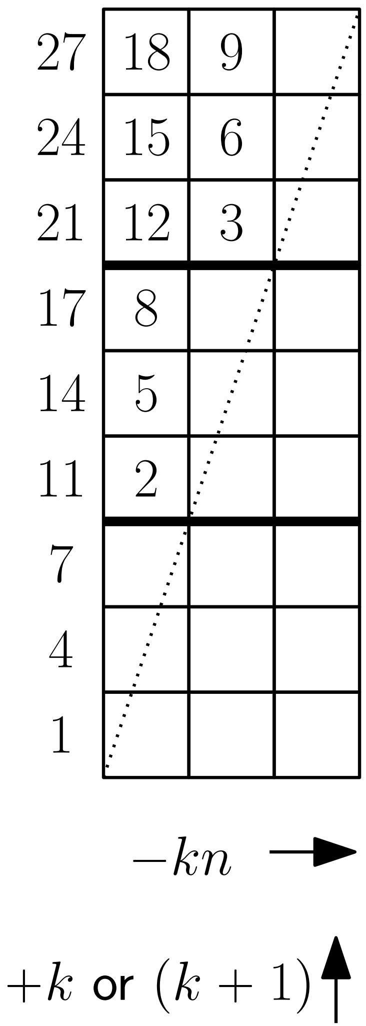
Bonus: affine permutations

- Word parking functions $\mathcal{WPF}_{K,k}$ can be translated into a set of affine permutations, which we call γ-**restricted**.

Rank function on cells:



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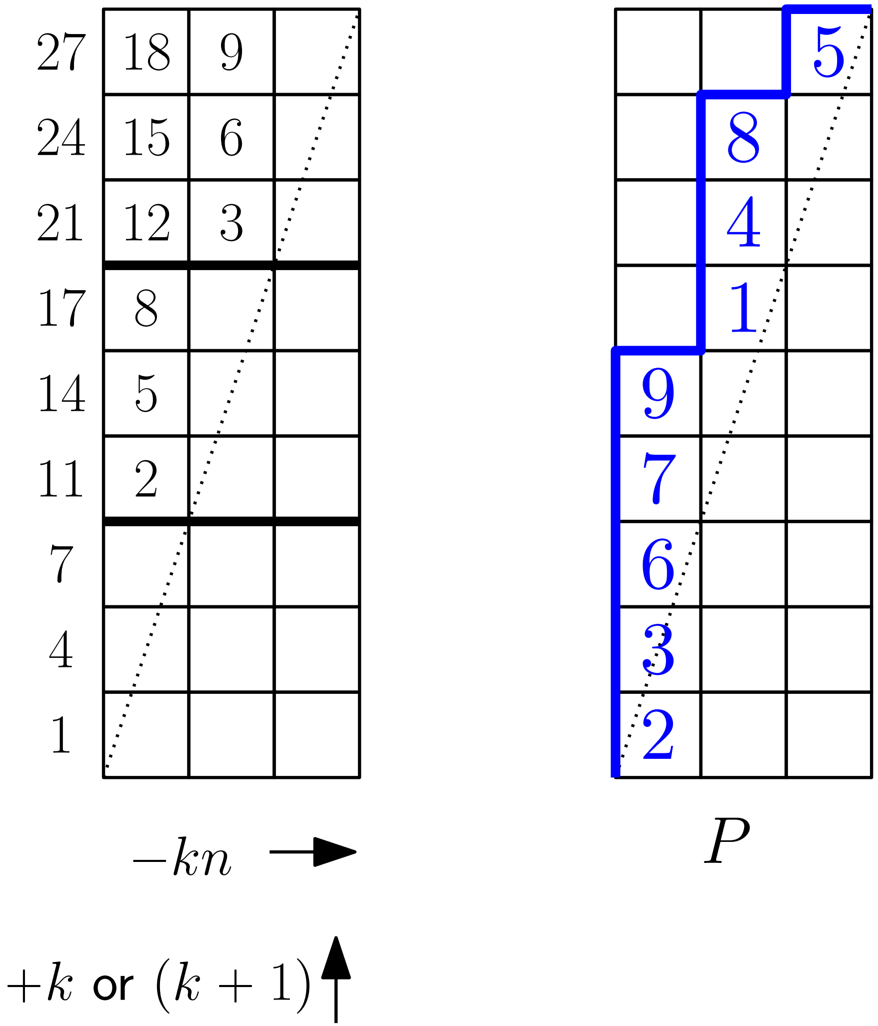

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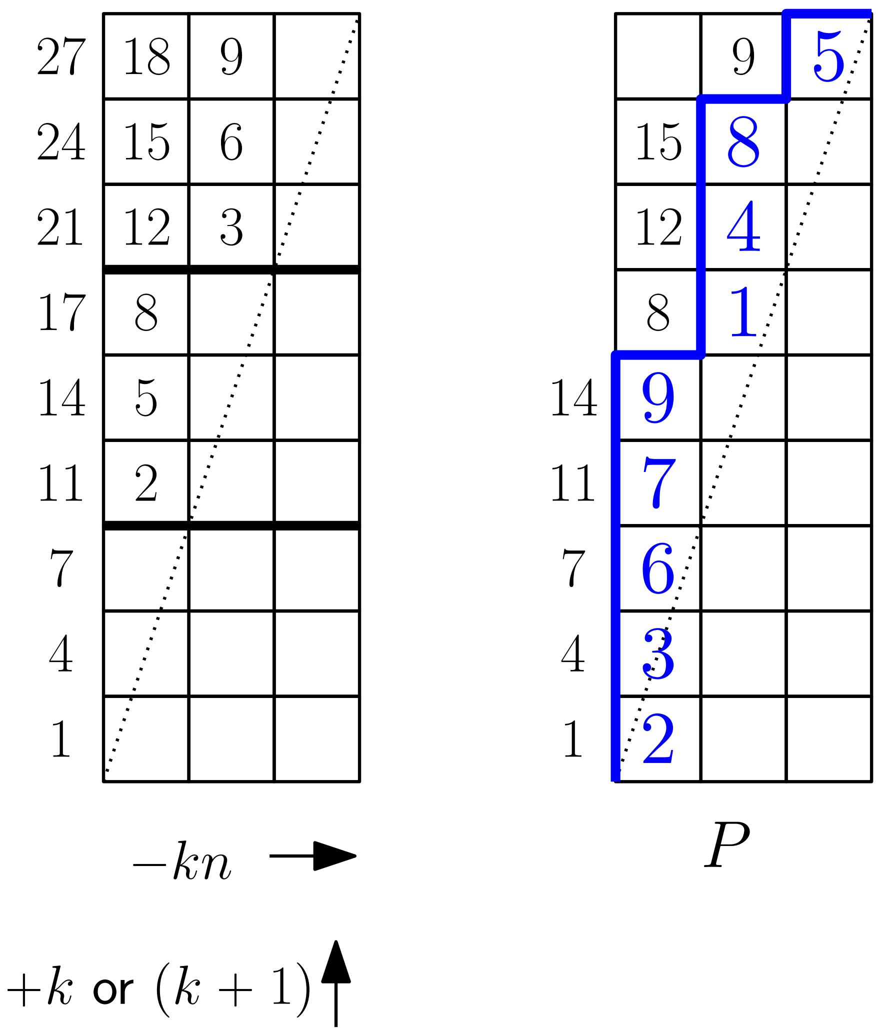

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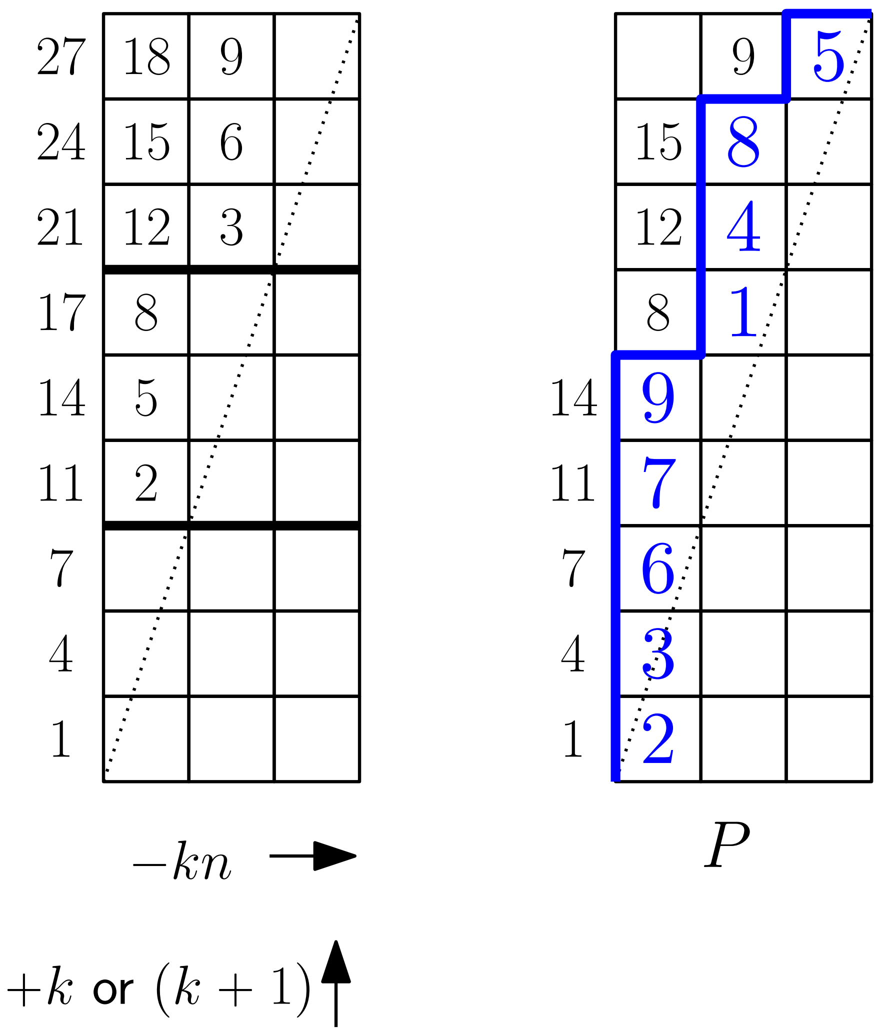


$w = [8, 1, 4, 12, 9, 7, 11, 15, 14] \in \widetilde{S}_9$

-w indexes a cell in $\widetilde{\mathrm{Fl}}_{\gamma}^{+,0}$

-The codim of the cell is $\mathrm{dinv}(P)$.

Delta Theorem

A different generalization of the Shuffle Theorem.

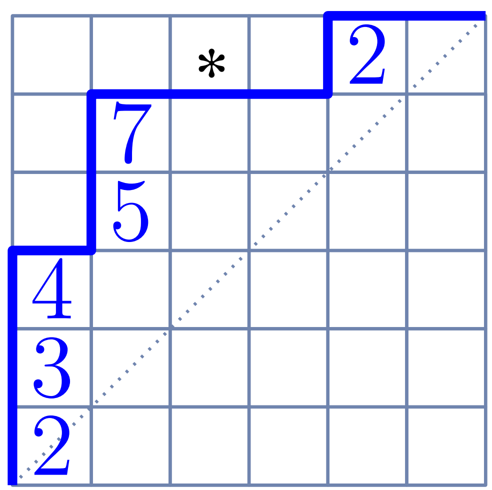
(Fall) Delta Theorem (D'Adderio–Mellit + BHMPS)

$$\Delta'_{e_{k-1}}(e_n) = \sum_{P \in \mathcal{WPF}^{\mathsf{fall}}_{n,k}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}^{-}(P)} x^P$$

When $n = k$, $\nabla e_n = \Delta'_{e_{n-1}} e_n$ and we recover the Shuffle Theorem.

Ex: $n = 6, k = 5$

$P =$



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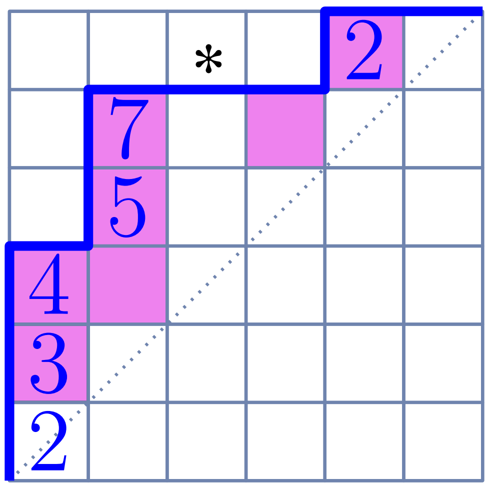

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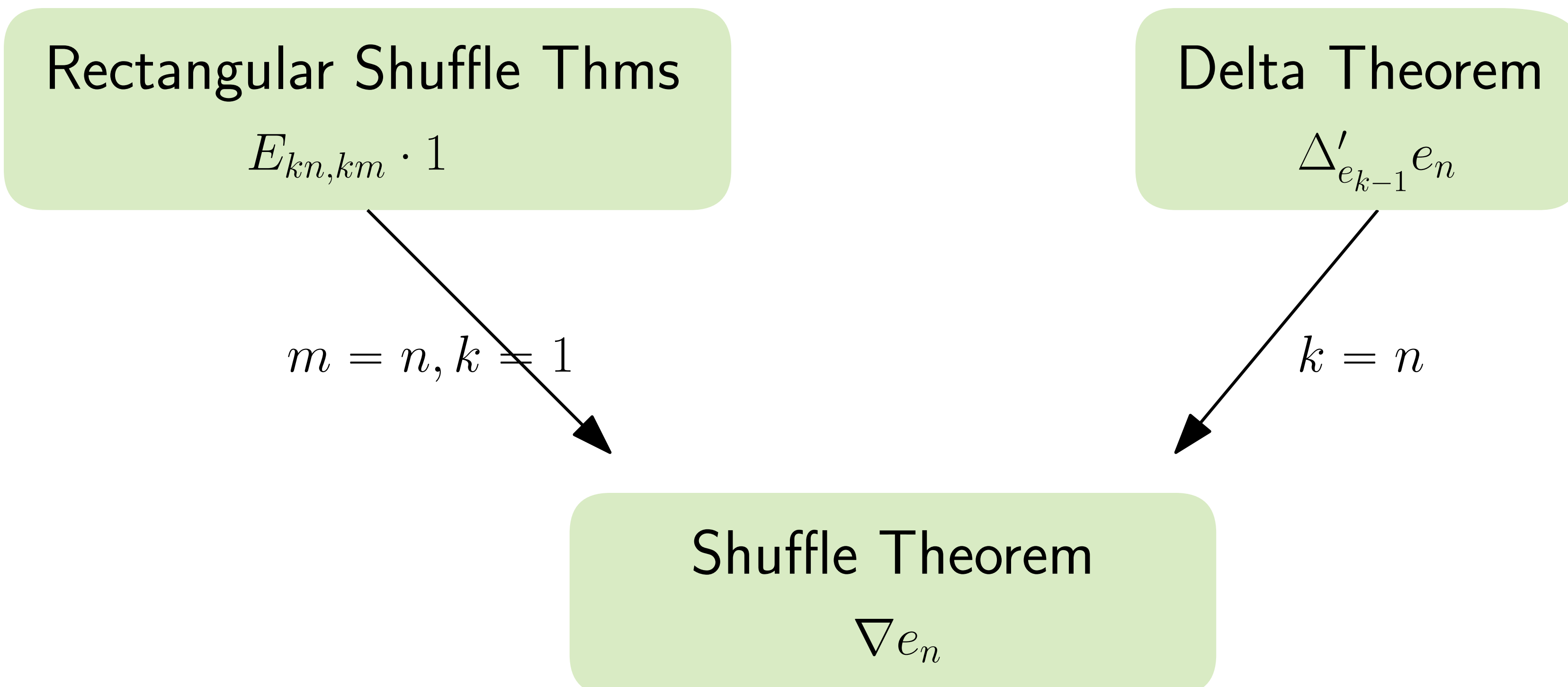
Rectangular Shuffle Thms
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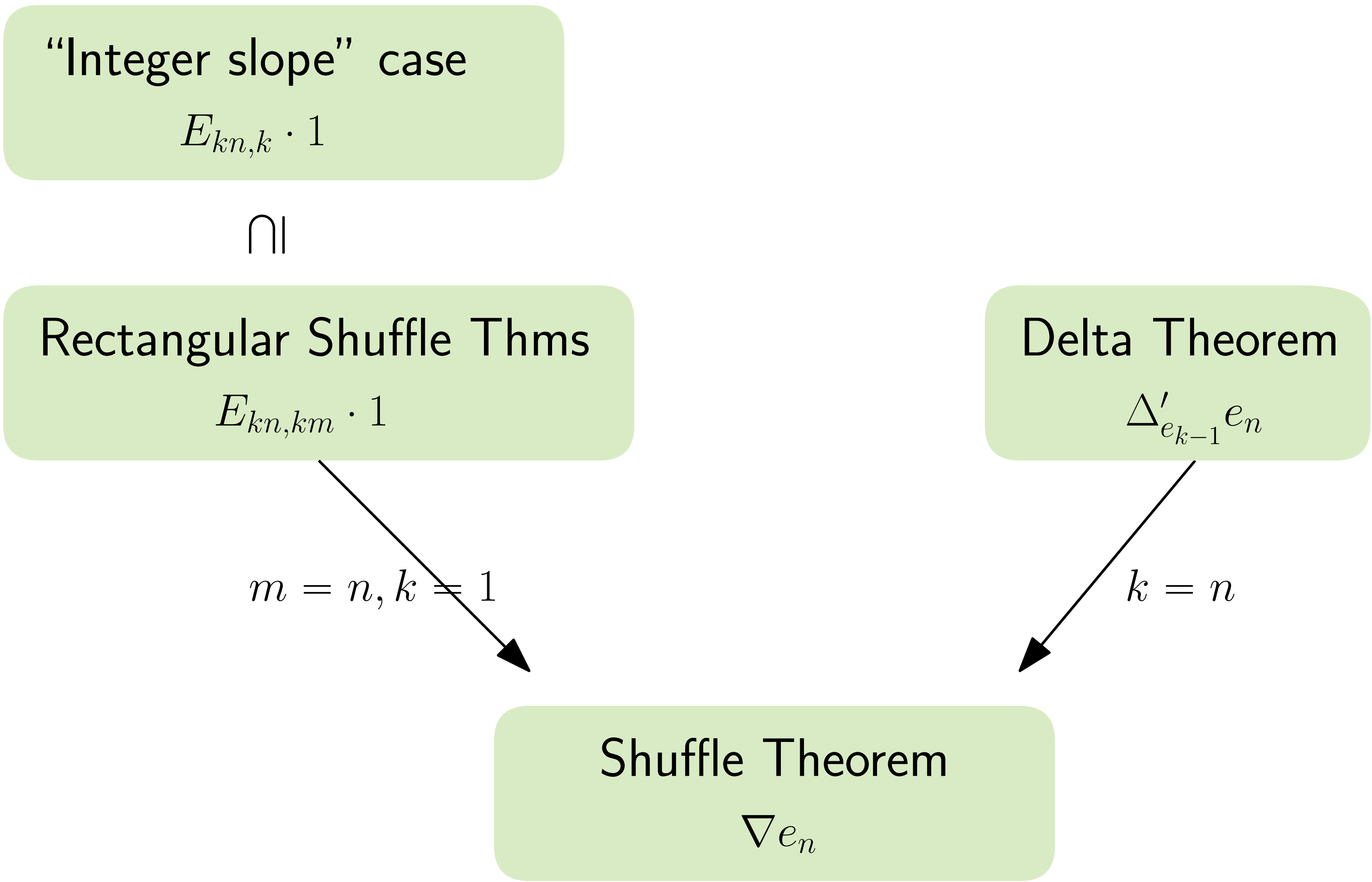
$m = n, k = 1$
$k = n$
Shuffle Theorem
∇e_n



"Integer slope" case
$E_{kn,k} \cdot 1$
$\cap|$
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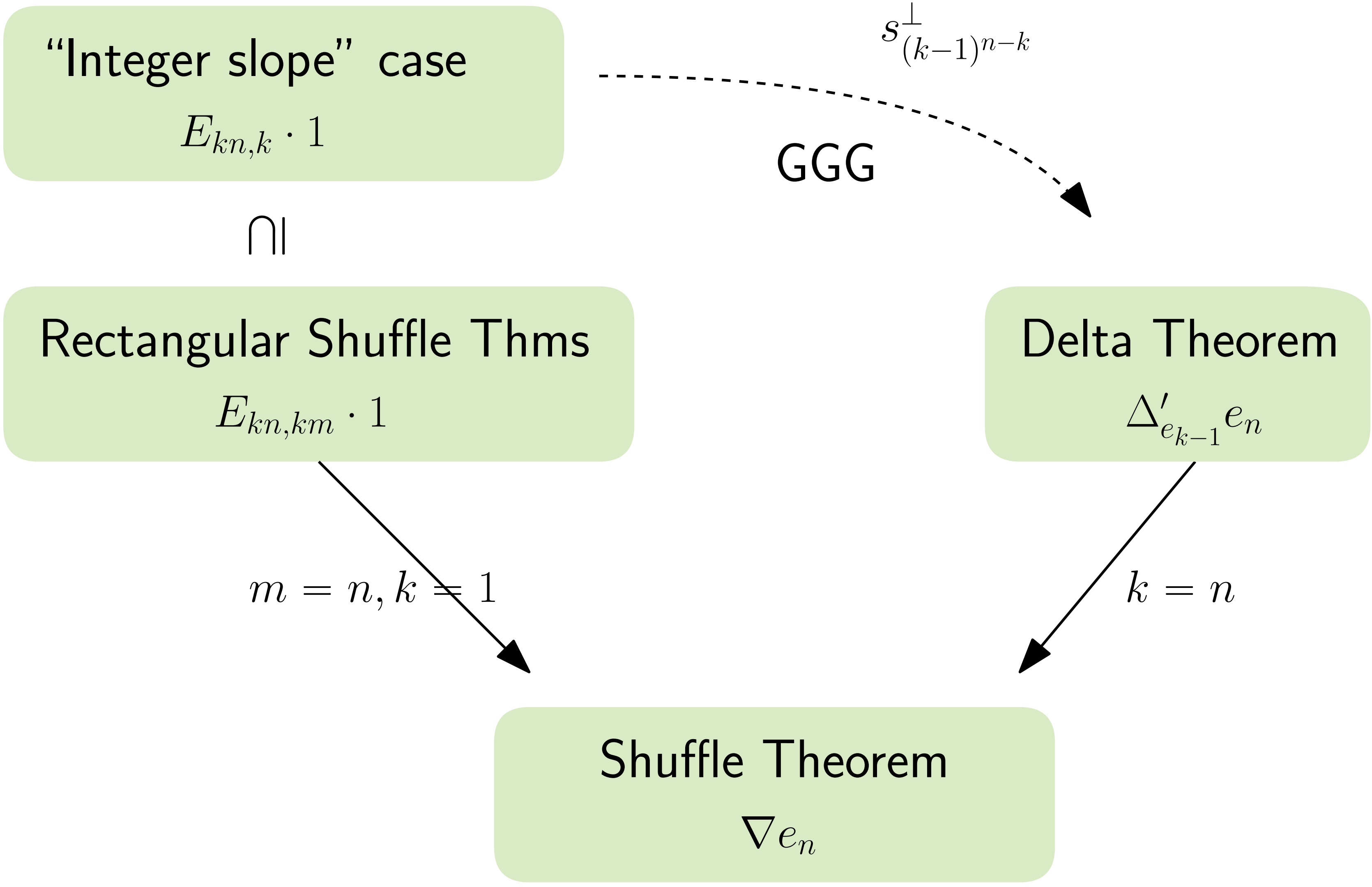

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GGG
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Skewing formula

The Delta Thm and Rectangular Shuffle Thm are **directly** related:

Theorem (Gillespie–Gorsky–G.)

Letting $\lambda = (k-1)^{n-k}$, then

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- The identity has geometric meaning in terms of affine Springer fibers.

Affine Springer fiber for Delta Thm

Take the same γ, $K = k(n-k+1)$

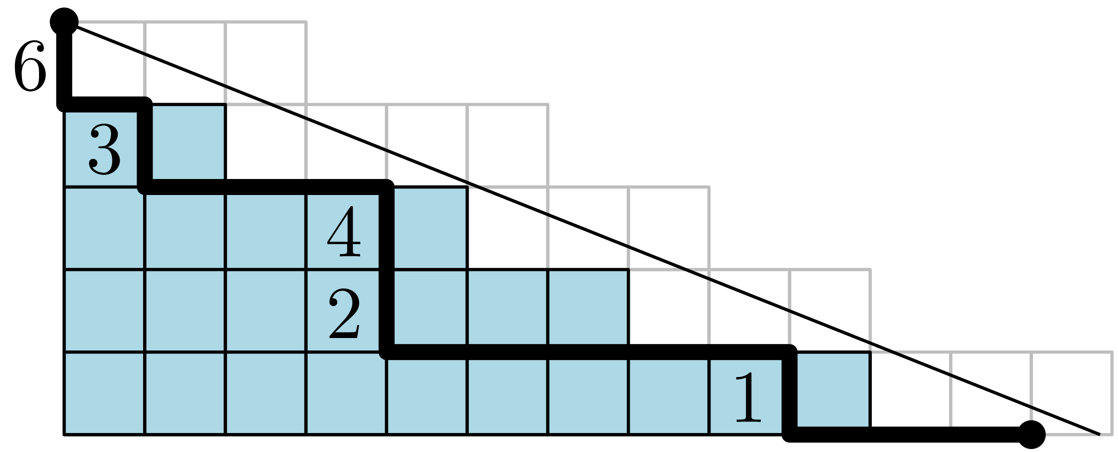
$$BM_\gamma := \{\Lambda_\bullet \in \widetilde{Fl}_{(K-n,1^n)} \mid \gamma\Lambda_i \subseteq \Lambda_i,\ JT(\gamma \circlearrowright \Lambda_0/\Lambda_1) \leqslant (n-k)^{k-1}\}.$$

Theorem (Gillespie–Gorsky–G.)

$$\mathrm{Frob}(H_*(BM_\gamma^{+,0};\mathbb{Q});q,t) = \Delta'_{e_{k-1}} e_n$$

Paths under any line

BHMPS have a Shuffle Theorem for paths under any line in the first quadrant.



Theorem (G, 2025)

If $b_1 \in \mathbb{Z}_{\geqslant 0}$, there is a γ such that

$$\mathrm{Frob}(H_*(\widetilde{Fl}_\gamma^{+,0}; \mathbb{Q}); q, t) = D_{b_1, b_2, \ldots, b_\ell} \cdot 1.$$

Summary

Operator	Parking func's	Springer fiber
∇e_n $\quad\leftrightsquigarrow$	$\mathcal{WPF}_n$ $\quad\leftrightsquigarrow$	$H_*(\widetilde{\mathrm{Fl}}_\gamma)$

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Thanks for listening!