Regular minimal indecomposable Hessenberg varieties

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Hessenberg varieties

Hessenberg varieties are subvarieties of the flag variety defined as follows.

Pick some $n \times n$ matrix X, and some **Hessenberg function** $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$, where we require $i \leq h(i)$ and $h(i) \leq h(i+1)$.

Define

$$Y_{X,h} = \{V_\bullet \in G/B \mid XV_i \subseteq V_{h(i)} \forall i\}.$$

Lie-theoretic definition of Hessenberg varieties

Alternatively, let $H \subseteq \mathfrak{gl}_n$ be a **Hessenberg space**, a subspace closed under bracket with $\mathfrak{b}$. Define

$$Y_{X,H} = \{gB \in G/B \mid g^{-1}Xg \in H\}.$$

If H is the space of $n \times n$ matrices $[m_{ij}]$ where $m_{ij} = 0$ whenever $i > h(j)$, then $Y_{X,H} = Y_{X,h}$.

This talk is in type A, but all our results are true (with the same proofs) in arbitrary type.

This definition tells us what happens when we translate $Y_{X,h}$ by some element $g' \in G$:

$$g' \cdot Y_{X,h} = Y_{g'Xg'^{-1},h}.$$

This means Hessenberg varieties for conjugate X are translates of each other.

Cell decompositions

The operator X is **Schubert-compatible** if X is upper triangular with at most one nonzero off-diagonal entry in each row and column.

Tymoczko showed that, if X is Schubert-compatible, then the intersections $C_w \cap Y_{X,h}$ are an affine paving of $Y_{X,h}$, and she gave an algorithm for determining which intersections are nonempty and calculating their dimensions. (This was extended by Precup to most cases, including all regular ones, in all types.)

This means $[\overline{C_w \cap Y_{X,h}}]$ is a basis for $H_*(Y_{X,h})$, and, when Poincaré duality holds, for $H^*(Y_{X,h})$.

Hessenberg-Schubert varieties

We call $C_w \cap Y_{X,h}$ a **Hessenberg–Schubert cell** and $Z_{w,X,h} := \overline{C_w \cap Y_{X,h}}$ a **Hessenberg–Schubert variety**.

Goal: Understand the cohomology ring $H^*(Y_{X,h})$, particularly if X is regular semisimple.

Approach: Understand the classes $[Z_{w,X,h}] \in H^*_T(G/B)$. In the GKM picture, this gives the T-multiplicity of $Z_{w,X,h}$ at every fixed point, which gives $[Z_{w,X,h}] \in H^*_T(Y_{X,h})$ (in the GKM picture).

Regular minimal decomposable Hessenberg varieties

A Hessenberg variety is **regular** if X is a regular operator - all of its Jordan blocks have different eigenvalues.

A Hessenberg variety is **minimal indecomposable** if $h(i) = i + 1$ for all $i < n$, or equivalently if

$$H = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha .$$

The permutahedral variety is the case where X is regular semisimple, and the Peterson variety where X is regular nilpotent.

For simplicity, we assume X is in Jordan form.

Cell combinatorics

In the regular minimal indecomposable Jordan form case:

- $C_w \cap Y_{X,h}$ is empty iff w has a long inverse descent within a Jordan block. This means $X_{i,i+1} = 1$ and $w^1(i) - w^{-1}(i+1) > 1$ (i shows up more than 1 place to the right of $i+1$). We say w is **admissible** (with respect to X) if $C_w \cap Y_{X,h}$ is NONempty.
- If w is admissible, then $C_w \cap Y_{X,h} \cong \mathbb{C}^d$, where d is the number of (usual) descents of w.

For X regular nilpotent, w is admissible iff $w = w_0^P$ for some P; for X regular semisimple, all w are admissible.

Cell example, I

Let

$$X = \begin{bmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}$$

with $\alpha \neq \beta$ (so X is regular of Jordan type $(3, 2)$).

If $w = 34215$, $C_w \cap Y_{X,h} = \emptyset$ because the 2 and the 3 are in the same Jordan block of X and form a long inverse descent.

Cell example, II

If $w = 41523$, $C_w \cap Y_{X,h} = \mathbb{C}^2$. We have

$$C_w = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & d & 1 & 0 \\ c & 0 & e & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $C_w \cap Y_{X,h}$ is the set where $b = c = e = 0$.

(In general, the conditions for being in $Y_{X,h}$ are more complicated and not even linear. One can inductively construct a parameterization from the definition of the Hessenberg variety.)

Translating the cell example

Observe that, if $v = 13524$, then

$$v \cdot (C_w) = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b & 0 & d & 1 & 0 \\ c & 0 & e & 0 & 1 \end{bmatrix},$$

In $C_w \cap Y_{X,h}$, $b = c = e = 0$, so

$$v \cdot (C_w \cap Y_{X,h}) = C_{vw}.$$

Descent parabolic decomposition

Given a permutation w, let P be the parabolic subgroup of S_n generated by the descents of w, so $s_i \in P$ iff $w(i) > w(i+1)$. For $w = 41523$, $P = S_2 \times S_2 \times S_1$.

We can define v by requiring $vw = w_0^P$. This makes v^{-1} a minimal length coset representative for P.

Main lemma

Lemma

Given any admissible w, with v and w_0^P defined as before,

$$v \cdot (C_w \cap Y_{X,h}) = C_{w_0^P} \cap Y_{v^{-1}Xv,h}$$

This tells us

$$Z_{w,X,h} = v^{-1} \cdot (Z_{w_0^P, v^{-1}Xv,h}).$$

$X_{w_0^P} = \overline{C_{w_0^P}}$ is a product of smaller flag varieties, and

$$Z_{w_0^P, v^{-1}Xv,h} = X_{w_0^P} \cap Y_{v^{-1}Xv,h}$$

is a product of smaller regular minimal indecomposable Hessenberg varieties (because they are irreducible).

Combinatorial description of closures

This tells us everything about the closures of cells. In particular,

- $Z_{w,X,h} \cap C_u$ is nonempty if and only if $u \in wW_P$ and $C_u \cap Y_{X,h}$ is nonempty. (We get u by moving around the entries in a descent block of w.)
- $\dim(Z_{w,X,h} \cap C_u)$ is the number of descents of u within a descent block of w.

Cell closure example

For $w = 41523$, X regular of Jordan type $(3,2)$, the closure of $C_w \cap Y_{X,h}$ intersects

- C_{14523} in dimension 1
- C_{41253} in dimension 1 (but $C_{41253} \cap Y_{X,h}$ has dimension 2).
- C_{14253} in dimension 0 (but $C_{14253} \cap Y_{X,h}$ has dimension 2).

Cohomology consequences

Translation doesn't change (ordinary) cohomology classes, and the Anderson–Tymoczko formula starting with knowing that

$$[X_{w_0^P}] = \frac{|W_P|}{|W|} \prod_{t_{ij} \notin W_P} (x_i - x_j)$$

gives

$$[\overline{C_w \cap Y_{X,h}}] = \frac{|W_P|}{|W|} \prod_{i+1<j} (x_i - x_j) \prod_{i:w(i)<w(i+1)} (x_i - x_{i+1}) \in H^*(G/B).$$

Here, P is the parabolic generated by the descents of w.

This is independent of the Jordan type of X (and depends only on the descents of w, as long as w is admissible with respect to X).

Poincaré duality for the Peterson variety

In the case where X is regular nilpotent, w is admissible if and only if $w = w_0^P$ for some parabolic P. The map $i_* : H_*(Y_{X,h}) \to H_*(G/B)$ is injective, and we see that, in $H_*(Y_{X,h})$,

$$i^* \left(\frac{|W_P|}{|W|} \prod_{i \notin P} (x_i - x_{i+1}) \right) \cap [Y_{X,h}] = [Z_{w_0^P, X, h}].$$

This means the geometric classes are rational multiples of squarefree monomials in the Klyachko generators $x_i - x_{i+1}$.

Singularities

We can tell which permutation flags are singular points.

Let Q be the parabolic generated by the Jordan blocks of X.

If Qw is a right coset that keeps the Jordan blocks together (and just reorders them), then w is singular in $Y_{X,h}$ according to the rules for Petersons worked out by Insko–Yong. (The top 3 cells are smooth; the rest singular.) (For other types, we also work out the singularity rules for Petersons.)

If Qw is not one of these right cosets, w is automatically singular.

Singularity example

X has Jordan type $(3,1)$

- $\{4123, 4213, 4132, 4321\}$ form a Peterson coset. The entire cell $C_{4123} \cap Y_{X,h}$ is singular; the others are nonsingular.
- $\{1432, 1423\}$ form a non-Peterson coset. Each cell has a 1-dimensional singular locus.
- $\{1243, 2143\}$ form a non-Peterson coset. Only the fixed point of $C_{1243} \cap Y_{X,h}$ is singular.
- $\{1432, 1423, 1243, 1234\}$ form a Peterson coset. C_{1234} is an isolated singular point.
- Summary: The boundaries of $C_{4213} \cap Y_{X,h}$ and $C_{4132} \cap Y_{X,h}$ are singular, plus the isolated point.

Thank you

Thank you for your attention!